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Entanglement Measures from Quantum Field Theory Methods in 1+1 Dimensions

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Quantum Matter, Benasque, Aragón (Spain)
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- This talk is mainly based on the papers:

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  A good review of our work can be found at:
  O. C-A and Benjamin Doyon, *Bi-partite entanglement entropy in massive (1+1)-dimensional quantum field theories*, J. Phys. A42 (2009) 504006; arXiv:0906.2946

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- What provides a good measure of entanglement? [Plenio & Virmani'05]
- The bi-partite entanglement entropy [Bennett et al.'96] and the logarithmic negativity [Peres'96; Eisert'00; Vidal & Werner'01; Plenio'05] are good measures of entanglement according to these properties

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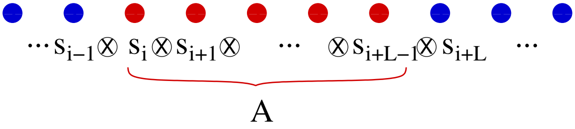

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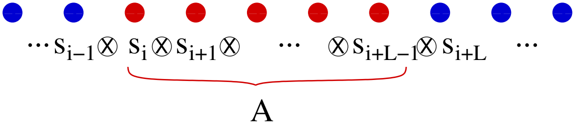

then we define

Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

$|\Psi\rangle$ ground state and ρ_A the reduced density matrix.

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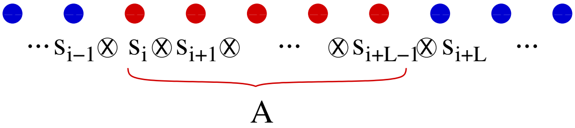

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- Other entropies may also be defined such as

Other Entropies

$$S_A^{\text{Rényi}} = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}, \qquad S_A^{\text{Tsallis}} = \frac{1-\mathrm{Tr}_A(\rho_A^n)}{n-1}$$

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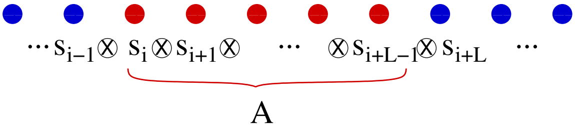

Replica Trick

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_A(\rho_A^n)$$

- For general QFTs the "replica trick" naturally leads to the notion of replica theories on multi-sheeted Riemann surfaces $\Rightarrow$ interpretation of $\mathrm{Tr}_A(\rho_A^n)$

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Logarithmic Negativity

$$\mathcal{E} = \log \mathrm{Tr}_{A\cup B}|\rho_{A\cup B}^{T_B}| \quad \text{with} \quad \rho_{A\cup B} = \mathrm{Tr}_C(|\Psi\rangle\langle\Psi|)$$

- Where $\mathrm{Tr}|\rho|$ represents the sum of the absolute values of the eigenvalues of ρ and T_B represents "partial transposition" in sub-system B

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- $|\Psi\rangle$ is the state of the whole system (for pure states)

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- The EE is a good measure of entanglement for pure states. What about mixed states?
- The LN provides a good measure of entanglement in mixed states for non-complementary regions such as A and B [Peres'96; Eisert'00; Vidal & Werner'01; Plenio'05]

- There is also a "replica" approach to the computation of the negativity [Calabrese, Cardy & Tonni'12]:

Logarithmic Negativity from the Replica Trick

$$\mathcal{E}[n] = \log \mathrm{Tr}_{A\cup B}(\rho_{A\cup B}^{T_B})^n \quad \text{then} \quad \mathcal{E} = \lim_{n\to 1} \mathcal{E}_e[n]$$

where $\mathcal{E}_e[n]$ means the function $\mathcal{E}[n]$ for n even. This limit requires analytic continuation from n even to $n = 1$

What QFT information is contained in the EE?

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Universal scaling: [Holzhey, Larsen & Wilczek'94; Vidal, Latorre, Rico & Kitaev'03; Calabrese & Cardy'04]:

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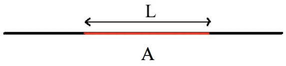

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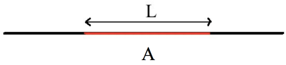

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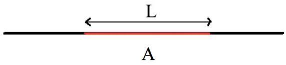

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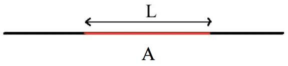

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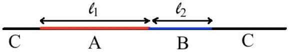

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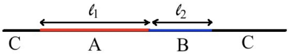

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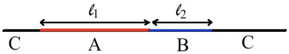

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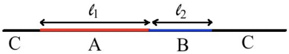

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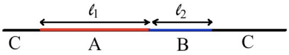

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$$\mathcal{E}^{\perp}(\ell) \sim -\frac{c}{4} \log m\epsilon + \mathcal{E}_{\text{sat}} - \frac{2a}{3\sqrt{3}\pi} K_0(\sqrt{3}m\ell)$$

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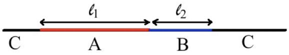

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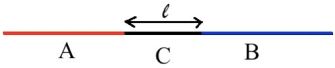
For semi-infinite non-adjacent regions:



$$\mathcal{E}^{\dashv\vdash}(\ell) \sim \frac{a(m\ell)^2}{2\pi^2} \left[K_0(m\ell)^2 + \frac{K_0(m\ell)K_1(m\ell)}{m\ell} - K_1(m\ell)^2 \right]$$

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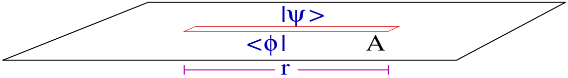
- QFT even allows us to access the universal part of the saturation constants (both for the EE and the LN) in the gapped regime
  $\Rightarrow$ The saturation constants U and $\mathcal{E}_{\rm sat}$ are related to expectation values of QFT fields and structure constants in CFT

So, how is this done?

Partition functions on multi-sheeted Riemann surfaces

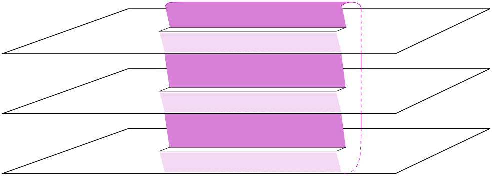
For integer number n of replicas, in the scaling limit, $\mathrm{Tr}_A(\rho_A^n)$ and $\mathrm{Tr}_{A\cup B}(\rho_{A\cup B}^{T_B})^n$ are partition functions on Riemann surfaces [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04; Calabrese, Cardy & Tonni'12]:

$_A\langle\phi|\rho_A|\psi\rangle_A \sim$



$$\mathrm{Tr}_A(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\ \mathcal{L}[\varphi](x)\right]$$

$\mathcal{M}_n =$



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- This is because, in essence, measures of entanglement (in the replica approach) are related to partition functions in Riemann manifolds whose complexity increases with the number of intervals involved
- If the theory is not critical (there is a mass scale) conformal invariance can not be used in any case so we need to find an alternative approach

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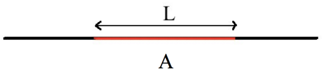

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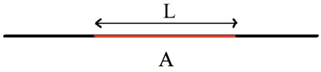

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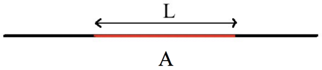

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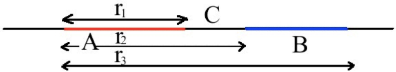

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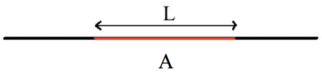

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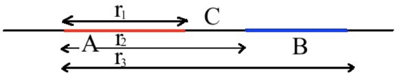

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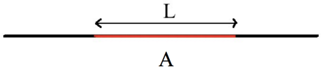

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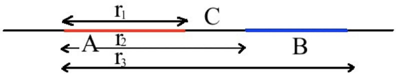

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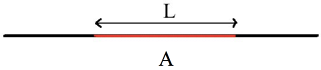

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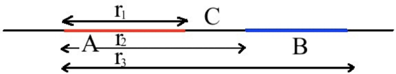

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Main Properties of Twist Fields

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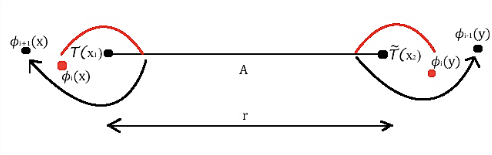

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