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Entanglement Entropy of the Lee-Yang Model from Branch Point Twist Fields

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This talk is based on ongoing (unpublished) work with D. Bianchini and B. Doyon

It builds on previous results for CFT which have been discussed by Benjamin Doyon in a previous talk and published in D. Bianchini, O.C.-A., B. Doyon, E. Levi and F. Ravanini, Entanglement Entropy of Non Unitary Conformal Field Theory, arXiv:1405.2804.

Twist Fields in QFT

- It has been known for some time that a “twist field” may be associated to the $\mathbb{Z}_n$ symmetry of an orbifolded CFT constructed as n cyclicly connected copies of a given CFT [Knizhnik’87]. The conformal dimension of such field $\mathcal{T}$ was also found by Knizhnik: $\Delta_{\mathcal{T}} = \frac{c}{24}\left(n - \frac{1}{n}\right)$.
- In the context of the investigation of the entanglement entropy a field of the same dimension was identified in [Calabrese, Cardy’04]. In this work, this field was interpreted as associated to a conical singularity in the complex plane.
- In 2008 we proposed [Cardy, O.C.-A. & Doyon’08] an interpretation of the fields found in [Calabrese & Cardy’04] as branch point twist fields.

Twist Fields in QFT

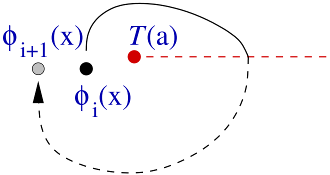
- Branch Point Twist Fields are characterized by the following commutation relations

$$\begin{aligned}\Psi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_{i+1}(y) \qquad x^1 > y^1,\\ \Psi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_i(y) \qquad x^1 < y^1,\end{aligned}$$

$$\begin{aligned}\Phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Phi_{i-1}(y) \qquad x^1 > y^1,\\ \Phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Phi_i(y) \qquad x^1 < y^1.\end{aligned}$$

  where Φ_i is a field of the original CFT living on copy i and $i = 1, \ldots, n$ and $n + i \equiv i$.

- Diagramatically:



EE as a Ratio of Correlation Functions

- The EE can be written as $-\lim_{n\to 1} \frac{\partial \mathrm{Tr}_A(\rho_A^n)}{\partial n}$ where [Calabrese & Cardy'04; Cardy, O.C.-A. & Doyon'08]

Entanglement Entropy in Unitary QFT

$$\mathrm{Tr}_A(\rho_A^n) \propto \epsilon^{\frac{c}{6}\left(n-\frac{1}{n}\right)} \langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle.$$

where ϵ is a short-distance cut-off, c is the effective central charge and r is the length of region A.

- For non unitary CFT we should rather consider

Entanglement Entropy vs Correlators

$$\mathrm{Tr}_A(\rho_A^n) \propto \epsilon^{\frac{c_{\mathit{eff}}}{6}\left(n-\frac{1}{n}\right)} \frac{\langle :\mathcal{T}\phi:(r) :\tilde{\mathcal{T}}\phi:(0)\rangle}{\langle \phi(r)\phi(0)\rangle^n}.$$

- The field $:\mathcal{T}\phi:$ is the leading term of the OPE of $\mathcal{T}$ and ϕ. It has conformal dimension $\Delta_{:\mathcal{T}\phi:} = \Delta_{:\tilde{\mathcal{T}}\phi:} = \Delta_{\mathcal{T}} + \frac{\Delta_\phi}{n}$
- ϕ is the primary field of lowest conformal dimension

Generalization to Massive QFT: the Lee-Yang Model

- We now want to compute the EE for a simple massive quantum field theory. The ideal model to look at is the Lee-Yang theory with S-matrix [Cardy & Mussardo'89]

$$S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)}.$$

- The underlying CFT is a minimal model with $c = -22/5$ and only one primary field (besides the identity). We call this field ϕ with $\Delta_\phi = -1/5$. This gives $c_{\text{eff}} = 2/5$.
- Form factors of the field ϕ in QFT were computed in [Zamolodchikov'91]. He was then able to compute $\langle \phi(r)\phi(0) \rangle$ with great precision and to match results to a perturbed CFT computation.

Form Factor Expansion

- Given a local operator $\mathcal{O}$ of a 1+1-dimensional QFT the two-point function

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \int \frac{d\theta_1}{2\pi} \cdots \int \frac{d\theta_k}{2\pi} |F_k^{\mathcal{O}}(\theta_1,\ldots,\theta_k)|^2 e^{-ix^\mu \sum_{i=1}^{k} p_\mu(\theta_i)}$$

where $F_k^{\mathcal{O}}(\theta_1,\ldots,\theta_k) := \langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_k\rangle$ is the k-particle form factor of $\mathcal{O}$.

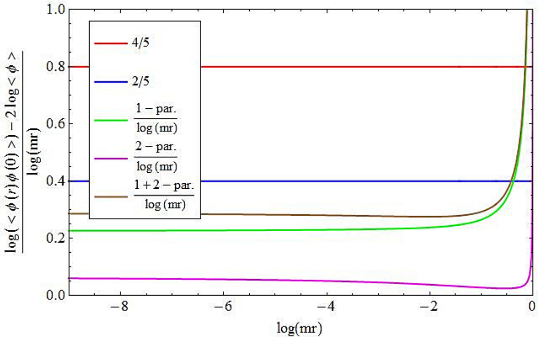
- In order to study the short distance behaviour it is more convenient to compute

$$\log(\frac{\langle \mathcal{O}(x)\mathcal{O}(0)\rangle}{\langle \mathcal{O}\rangle^2}) = \sum_{k=1}^{\infty} \frac{1}{k!} \int \frac{d\theta_1}{2\pi} \cdots \int \frac{d\theta_k}{2\pi} H_k^{\mathcal{O}}(\theta_1,\ldots,\theta_k) e^{-ix^\mu \sum_{i=1}^{k} p_\mu(\theta_i)}$$

where H_k are related to F_k by simply matching terms in the two formulae.

Two-Point Functions of Non-Unitary Theories

- In fact, we can do some numerics using Zamolodchikov's form factors and compute $\log\left(\frac{\langle\phi(r)\phi(0)\rangle}{\langle\phi\rangle^2}\right)$ and we get:



- As expected, the logarithm of the two-point function is proportional to $\log(mr)$ at small distances but the proportionality constant is not $-4\Delta_\phi = 4/5$ but rather $-2\Delta_\phi = 2/5$.

Two-Point Functions of Non-Unitary Theories

- This behaviour was well understood in [Zamolodchikov'91]: at the critical point the leading OPE contribution scales as $r^{-4\Delta_\phi} = r^{4/5}$ but in the massive theory $\langle \phi \rangle \neq 0$ so that the leading contribution to the "perturbed OPE" does not come from the identity field but from ϕ itself. This produces the term $r^{-2\Delta_\phi} = r^{2/5}$.
- Interestingly, the same occurs to the correlator $\langle : \mathcal{T}\phi : (r) : \tilde{\mathcal{T}}\phi : (0) \rangle$ in the massive theory. Its leading behaviour in perturbed CFT will be given by a term proportional to $\langle \phi \rangle^n \neq 0$ corresponding to the field $\phi_1 \cdots \phi_n$. Thus $\langle : \mathcal{T}\phi : (r) : \tilde{\mathcal{T}}\phi : (0) \rangle \propto (rm)^{-4\Delta_{:\mathcal{T}\phi:} - 2n\Delta_\phi}$.
- Crucially, when these two behaviours are combined together the EE of QFT still displays the correct behaviour (predicted by CFT) at short distances.

1 and 2-Particle Form Factors of Twist Fields

- We need to solve the twist field form factor equations given in [Cardy, O.C.-A. & Doyon'08]. For the 1- and 2-particle form factors they can be summarized as

$$F_2^{ab}(\theta) = S_{ab}(\theta) F_2^{ba}(-\theta) = F_2^{b\,a+1}(-\theta + 2\pi i)$$

with $S_{ab}(\theta) = (S(\theta))^{\delta_{ab}}$,a,b $\in [1, n]$ and

$$\underset{\theta=\bar{\theta}}{\text{Res}} \; F_2^{aa}(\theta - \bar{\theta} + \frac{2\pi i}{3}) = i\Gamma F_1^a$$

where

$$\Gamma^2 = -i \lim_{\theta \to \frac{2\pi i}{3}} (\theta - \frac{2\pi i}{3}) S(\theta) = -2\sqrt{3}.$$

1 and 2-Particle Form Factors of Twist Fields

- We need to compute the form factors (FFs) of $:\mathcal{T}\phi:$. For $n=1$ they should reduce to those of ϕ. In contrast, the form factors of $\mathcal{T}$ should vanish at $n=1$ (except for $\langle\mathcal{T}\rangle$ as $\mathcal{T}$ becomes the identity).
- Putting all these conditions together we find:

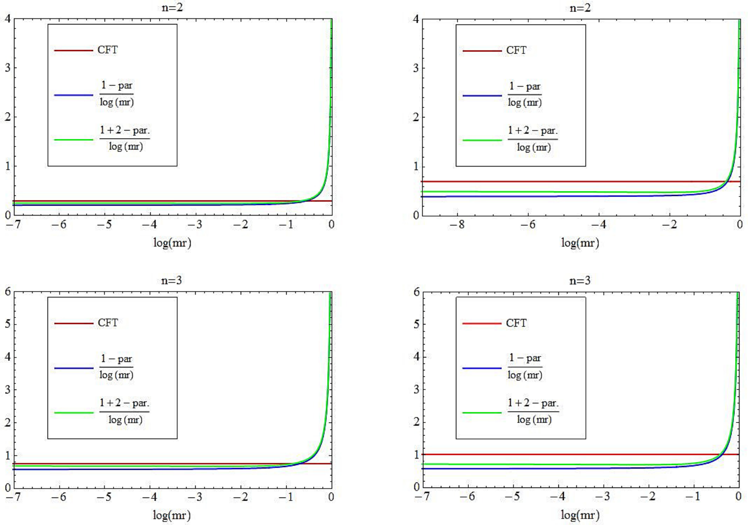
$$F_1^1 = \frac{\Gamma\langle\mathcal{O}\rangle\left(1\pm\sqrt{1-\frac{2\sin\left(\frac{\pi}{6n}\right)\cos\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{3n}\right)\cos^2\left(\frac{\pi}{3n}\right)}}\right)}{2n\tan\left(\frac{\pi}{3n}\right)f\left(\frac{2i\pi}{3},n\right)}$$

$$F_2^{11}(\theta) = \frac{\langle\mathcal{O}\rangle\sin\left(\frac{\pi}{n}\right)}{2n\sinh\left(\frac{i\pi-\theta}{2n}\right)\sinh\left(\frac{i\pi+\theta}{2n}\right)}\frac{F_{\rm min}^{11}(\theta,n)}{F_{\rm min}^{11}(i\pi,n)}+\frac{(F_1^1)^2}{\langle\mathcal{O}\rangle}F_{\rm min}^{11}(\theta,n)$$

where $F_{\rm min}^{11}(\theta)$ is a minimal solution to the FF equations which is proportional to a known function $f(\theta,n)$.
- Setting $n=1$ gives either $F_1^1=0$ or $F_1^1=F_1^\phi$. This is strong indication that the FFs do indeed correspond to $\mathcal{O}=\mathcal{T}$ and $\mathcal{O}=:\mathcal{T}\phi:$.

Numerical Evidence

The figures represent $\frac{\log(\langle \mathcal{O}(r)\tilde{\mathcal{O}}(0)\rangle) - 2\log\langle \mathcal{O}\rangle}{\log(mr)}$ for $\mathcal{O} = \mathcal{T}$ and for $\mathcal{O} =: \mathcal{T}\phi :$.



Next-to-leading order correction to EE of large regions

If we now consider

$$S = -\lim_{n\to 1}\frac{\partial}{\partial n}\epsilon^{\frac{c_{\mathrm{eff}}}{6}\left(n-\frac{1}{n}\right)}\frac{\langle :\mathcal{T}\phi:(r):\tilde{\mathcal{T}}\phi:(0)\rangle}{\langle\phi(r)\phi(0)\rangle^n}$$

and we use a FF expansion

$$= -\frac{c_{\mathrm{eff}}}{3}\log(m\epsilon) + U - \underbrace{\frac{2}{\pi f(\frac{2\pi i}{3},1)^2}\left(\frac{1}{\sqrt{3}}-\frac{13\pi}{108}\right)}_{0.0769782}K_0(mr)+\cdots$$

where

$$U = -\frac{\partial}{\partial n}\left(\frac{m^{4\Delta_{:\mathcal{T}\phi:}}\langle :\mathcal{T}\phi:\rangle^2}{m^{4n\Delta_\phi}\langle\phi\rangle^{2n}}\right)_{n=1}$$

This means that the next-to-leading order correction to the EE is different in non-unitary models and determined by the 1-particle FF!

Conclusions

- We have shown that the EE of non-unitary QFTs admits an analogous expression to that predicted by CFT.
- We have tested this expression numerically and shown that the CFT behaviour is recovered for small regions (short distances).
- We have found that for large regions (distances) the EE saturates as for unitary models but the next-to-leading correction to saturation is different from that found for unitary 1+1-dimensional models.
- It would be very interesting to test this prediction numerically but studying the quantum spin chain described in Benjamin's talk.