CITY UNIVERSITY LONDON

Entanglement Entropy in 1+1 Dimensional QFT

Olalla A. Castro-Alvaredo

School of Mathematics, Computer Science and Engineering
Department of Mathematics
City University London

Department of Mathematics, Brunel University
March 2015

My background

Since I became research active in 1999, my research has centered around integrable 1+1 dimensional quantum field theories.

They have been studied for a long time because of their nice properties: non-perturbative calculations are possible!

In particular much work has been carried out focussing of the "form factor programme". That is, computing matrix elements of the form

$$\langle 0|\mathcal{O}(x)|n\rangle,$$

where $\langle 0|$ is the ground state and $|n\rangle$ is an n-particle state.

The bi-partite entanglement entropy is related to correlation functions (both in QFT and also spin chain systems).

Overview of the talk

- Entanglement entropy
- Motivation
- Twist Fields and Riemann Surfaces
- Entropy and Twist Fields
- Large distance behaviour of the entropy
- Numerical tests
- Conclusions and Outlook

Our contribution so far...

D. Bianchini, OC-A, B. Doyon, arXiv:1502.03275
D. Bianchini, OC-A, B. Doyon, E. Levi and F. Ravanini, J. Phys. A48 04FT01 (2015) (IoP Select)
E. Levi, OC-A and B. Doyon, Phys. Rev. B 88 094439 (2013)
OC-A and B. Doyon, J. Stat. Mech. P02016 (2013)
OC-A and B. Doyon, Phys. Rev. Lett. 108 120401 (2012)
OC-A, B. Doyon and E. Levi, J. Phys. A 44 492003 (2011)
OC-A and E. Levi, J. Phys. A 44 255401 (2011) (IoP Select)
OC-A and B. Doyon, J. Stat. Mech. P02001 (2011)
OC-A and B. Doyon, J. Stat. Phys 134(1) 105 (2009)
OC-A and B. Doyon, J. Phys. A 42 504006 (2009)
B. Doyon, Phys. Rev. Lett. 102 031602 (2009)
J.L. Cardy, OC-A and B. Doyon, J. Stat. Phys. 130(1) 129 (2008)

Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.
  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \;, \quad \langle \hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by **local variables**.
- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

$$|\hat{\psi}\rangle = \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

$$= \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right)$$

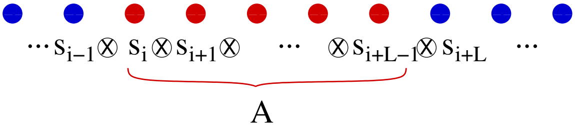
- This is particular to **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ , \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

  (for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).
- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement?

Bi-partite Entanglement Entropy

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$



then we define

Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

Replica Trick

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$$

$|\Psi\rangle$ ground state and ρ_A the reduced density matrix.

- For general QFTs the "replica trick" naturally leads to the

Motivation

The study of the EE of extended quantum systems is a popular area of research in various areas:

- **Quantum Information:** The EE quantifies the amount of “surprise” that a sub-part of a system finds when discovering it is correlated to the rest of the system. Therefore, entanglement entropy is a *bona fide* measure of the correlations in the system [Latorre & Riera, Review’09]
- **Quantum Field Theory:** The EE can be defined for any QFT (operator independent). It provides “universal” information about quantum systems/quantum states. [Callan & Wilczek ’94; Holzhey, Larsen & Wilczek ’94; Latorre, Rico & Kitaev’03; Latorre, Rico & Vidal’04; Calabrese & Cardy ’04; J.L. Cardy, O.C-A & B. Doyon’08]
- **Holography (AdS/CFT Correspondence):** The EE in the CFT is given by the area of a certain extremal surface in the bulk (AdS) [Ryu and Takayanagi’06]

Motivation in QFT

- It is a theoretical measure of entanglement. It is a particular way of extracting information about the state of a quantum system.
- This information does not depend on the correlation functions of any fields (just on the state).
- Near critical points (e.g. Conformal Field Theory) it displays universal behaviour.
- In the context of high energy physics much of the motivation to study the entanglement entropy has come from its behaviour at quantum critical points [Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04]:

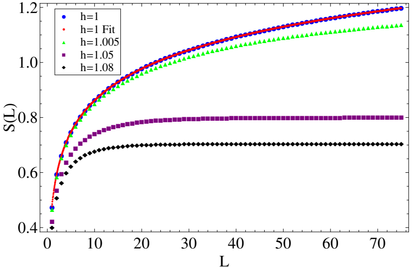
$$S_m \sim \frac{c}{3} \log m \quad \Rightarrow \quad \text{information about the CFT}$$

e.g. the EE of a subsystem of length m diverges logarithmically. The proportionality constant c is called the central change. It uniquely characterises the CFT.

Example: the Ising model

$$H = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + h\sigma_i^z\right)$$

- We may carry out the "scaling limit" of this theory in two different ways:
- Set $h = 1$ from the beginning: then $\xi = \infty$ and in the limit $N \to \infty$ this is a critical model.
- Take $h > 1$: $\xi \propto m^{-1}$ finite but large. Taking $N \to \infty$ while L/ξ is finite we obtain *Ising field theory*.



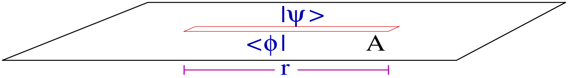
- $S(L) = \frac{0.500003}{3}\log L + 0.478551$ for $h = 1$. For $h > 1$ saturation is reached [Vidal, Latorre, Rico & Kitaev'03].

More motivation

- The entanglement entropy can reveal a great amount of information about the CFT (not just c). For example, the entropy of excited states [Ibañez Berganza, Castillo Alcaraz & Sierra'11] and the negativity [Cardy, Calabrese & Tonni'12] both encode information about the conformal dimensions of primary fields.
- In theories with boundaries, the entropy provides information about the degeneracy of the ground state (g-function)[Afleck & Ludwig'91; Calabrese & Cardy'04; OCA & Doyon'09].
- The entanglement entropy can also provide very interesting information about the state in cases when there is no conformal critical point [Ercolessi, Evangelisti, Franchini & Ravanini'10; Popkov & Salerno'04; Popkov, Salerno & Schütz'05], especially about the "geometry" of the state [OCA & Doyon'11;12].

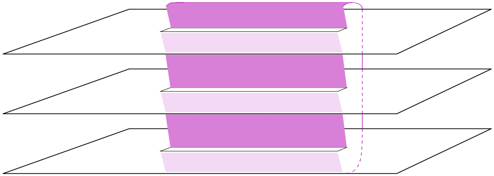
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04] ($\mathrm{Tr}_A(\rho_A)$ is the partition function of the original theory!):

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\; \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n =$$



How are the CFT results obtained?

- In CFT this "geometric" picture is particularly powerful because of conformal invariance.
- It is possible to use conformal maps and uniformization theorem to map the Riemann surface above to the complex plane with two conical singularities:

$$z = g(\omega) = \left(\frac{\omega - x_1}{\omega - x_2}\right)^{1/n} \quad \text{with} \quad z \in \mathbb{C}^2 \quad \text{and} \quad \omega \in \mathcal{M}_n$$

- It is possible to use this map to work out $\mathrm{Tr}_A(\rho_A^n)$. Then the EE can be computed.
- The problem with this approach is that it relies heavily on conformal symmetry. It is not applicable to systems which are not critical (e.g. have a mass scale).
- The is another point of view which allows for generalisation: employing twist fields.

Twist Fields I

- For general 1+1 dimensional QFT we have found [Cardy, OCA & Doyon'08] that the entropy may be expressed in terms of a two-point function of twist fields:

$$Z_n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

where D_n is a normalisation constant, and d_n is the scaling dimension of $\mathcal{T}$ [Knizhnik'87; Calabrese & Cardy'04]:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \sim r^{-2d_n} \ \Rightarrow \ S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \sim \langle \mathcal{T}\rangle_n^2 \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m\varepsilon) + U$$

Twist Fields II

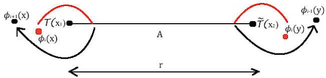
- The Twist Fields are defined through very general commutation relations with the fundamental field of the model:

$$\begin{aligned}\Psi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Psi_{i+1}(y) \qquad x^1 > y^1, \\ \Psi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Psi_i(y) \qquad x^1 < y^1,\end{aligned}$$

$$\begin{aligned}\Psi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Psi_{i-1}(y) \qquad x^1 > y^1, \\ \Psi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Psi_i(y) \qquad x^1 < y^1.\end{aligned}$$

for $i = 1, \ldots, n$ and $n + i \equiv i$.

- Diagramatically:



Entropy from Form Factors

- The two-point function of branch-point twist fields can be decomposed into the *in*-basis, giving a large-distance expansion:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathbb{R}^n} = \langle \text{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\text{vac}\rangle =$$

$$\sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k=1}^{n}\int \frac{d\theta_1\cdots d\theta_k}{(2\pi)^k}\,|F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)|^2 e^{-r\sum\limits_{i=1}^{k} m_{\mu_i}\cosh\theta_i}$$

where

$$F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k) = \langle \text{vac}|\mathcal{T}(0)|\theta_1,\ldots,\theta_k\rangle^{in}_{\mu_1,\ldots,\mu_k}$$

are the k-particle form factors of the twist-field $\mathcal{T}$.

- Typically the expansion is rapidly convergent in k for large r (short-distance expansion).
- These form factors can be computed as the solutions to a set of consistency equations which we formulated in our first paper [Cardy, OCA & Doyon'08].

Scaling behaviour of the von Neumann entropy

Let r be the length of region A in a massive 1+1 dimensional QFT, then

- Short distance: $0 \ll r \ll \xi$, *logarithmic divergency*

$$S(r) \sim \frac{c}{3} \log\left(\frac{r}{\varepsilon}\right) + c_2$$

- ε is a short distance cut-off and c_2 a non-universal constant.
- Large distance: $0 \ll \xi \ll r$, *saturation*

$$S(r) = -\frac{c}{3}\log(\varepsilon m_1) + U - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3rm_1}\right)$$

$$S(r) = \underbrace{-\frac{c}{3}\log(\varepsilon \mathbf{m_1}) + \mathbf{U}}_{\text{saturation}} - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3rm_1}\right)$$

- U is a universal constant which can be computed in QFT. m_1 is the mass of the lightest particle in the theory.

$$S(r) = -\frac{c}{3}\log(\varepsilon m_1) + U - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3rm_1}\right)$$

QFT from quantum spin chains: the scaling limit

- Consider now a quantum spin chain of length N, lattice spacing a and block length L.
- We will consider the thermodynamic and scaling limits when $N \to \infty$ and $a \to 0$, simultaneously.
- The spin chain we will look at (Ising model) has a critical point for $h = 1$. Taking $h \to 1$ followed by $L \to \infty$ we expect to recover the short distance behaviour ($\epsilon = a$).

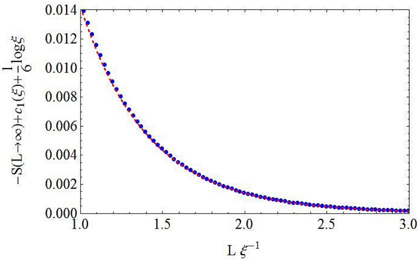
Entropy of Critical Chain

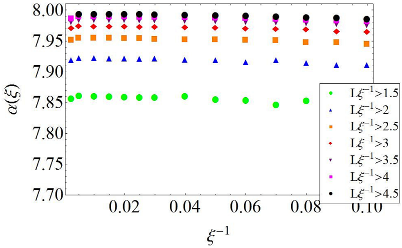
$$S(L) \sim \frac{c}{3} \log L + c_2$$

- Taking $h \to 1$ and $L \to \infty$ simultaneously, with $\frac{L}{\xi} =: m_1 r$ fixed, we should recover the QFT saturation behaviour

Entropy of Gapped Chain

$$S(L) \sim \frac{c}{3} \log(\xi) + c_1 - \frac{1}{8} K_0(2L/\xi)$$





- Numerical results are fitted to the function $1/8K_0(2L/\xi)$ with very good agreement.
- $\alpha(\xi)$ is the inverse of the coefficient of the Bessel function (expected to be 8 at the critical point). It approaches the value 8 for increasing ratios L/ξ.

Conclusions

- Quantum field theory techniques are a powerful tool for predicting the scaling behaviour of the entropy of both critical and non-critical systems (e.g. spin chains).
- Our numerics suggest that non-critical chains display the behaviour predicted by QFT

$$S(L) \approx \frac{c}{3} \log(\xi) + c_1 - \frac{a}{8} K_0(2L/\xi)$$

with $a =$ number of lightest particles for large blocks
- The EE encapsulates information about the particle spectrum of non-critical theories in 1+1 dimensions.
- Many open problems remain in this area which is still dominated by the investigation of critical systems.
- A natural next step is to look at other measures of entanglement which are more natural for mixed states (e.g. negativity), consider the EE of excited states and/or disconnected regions.