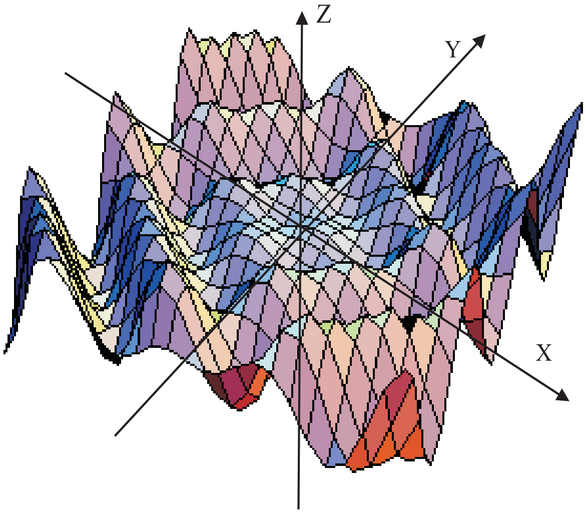
CITY UNIVERSITY
LONDON

NOTES FOR MA2601 CALCULUS

ACADEMIC YEAR 2009/10

http://www.staff.city.ac.uk/o.castro-alvaredo/teaching/calculus.html



Dr. Olalla A. Castro Alvaredo
Centre for Mathematical Science
Room C126
Tel.: 020 7040 8952
e-mail: o.castro-alvaredo@city.ac.uk

Surgery hours: Wed. from 11:00 to 12:30 and from 15:00 to 16:30

Reading List

Some suitable textbooks for this module are:

1. Calculus: A complete course, by R.A. Adams (Addison Wesley).
2. Calculus, by H. Anton (Wiley).
3. Calculus: One and several variables, by S.N. Salas and E. Hille (Wiley).
4. University Calculus, by J. Hass, M.D. Weir and G.B. Thomas (Addison Wesley).

All the books above are of a similar level and content. Any of them will be useful for the course and will cover all or the majority of the material we will see in the course.

Dr Olalla Castro Alvaredo will not follow any of the books above in particular, but will take parts of different books as inspiration for her lectures.

Dr Olalla Castro Alvaredo will also provide her own notes for the course, which are based upon some of the books above. This notes, together with the notes you should take during the lectures should be sufficient for the course.

Contents

1 Introduction: what this module is about

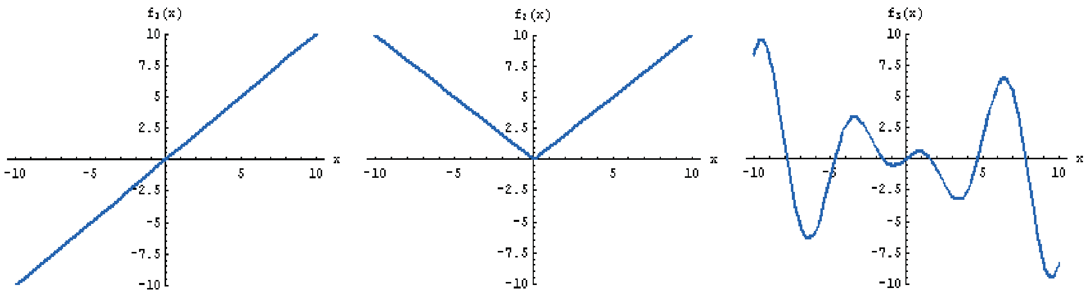
A large part of this course will be devoted to the extension of many definitions and operations you have learn in 1st year Calculus to the case of functions of several variables, concentrating specially on functions of two variables. Remember that a real function $f(x)$ of one real variable x is nothing but a rule of the following type

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\rightarrow f(x) \end{aligned} \tag{1.1}$$

namely, an operation which takes a real number $x \in \mathbb{R}$ as its input and gives another real number $f(x) \in \mathbb{R}$ as output. $\mathbb{R}$ denotes the set of all real numbers. For example the functions

$$f_1(x) = x; \;\; f_2(x) = |x|; \;\; f_3(x) = x\cos(x)\ldots \qquad \text{etc} \tag{1.2}$$

are all functions of one real variable x.

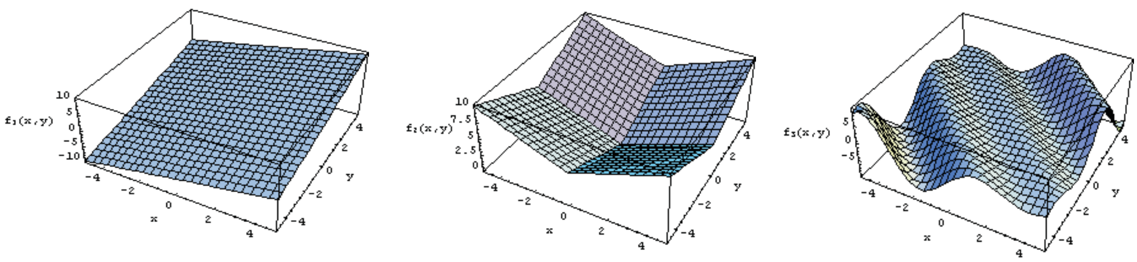


Similarly we can define real functions of two real variables $f(x, y)$ as maps

$$\begin{aligned} f : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x, y) &\rightarrow f(x, y) \end{aligned} \tag{1.3}$$

which take an element of $\mathbb{R}^2$ (that is, a point of the plane) $(x, y) \in \mathbb{R}^2$ as input and produce a real number $f(x, y) \in \mathbb{R}$ as output. For example,

$$f_1(x, y) = x + y; \;\; f_2(x, y) = |x| + |y|; \;\; f_3(x, y) = (x + y)\cos(x + y)\ldots \qquad \text{etc} \tag{1.4}$$



As you can see in the picture, a plot of a function of two real variables generates a surface in a 3D-space, whereas a plot of a function of one real variable generated a curve in the 2D-plane.

Last year you have studied various properties of functions of the type (1.1). Some of then are for instance the notions of

- continuity,

- differentiability,

and you have also learned techniques to compute

- limits (e.g. l'Hôpital limits),
- expansions around one point (e.g. Taylor series),
- integrals (e.g. integration by parts),
- derivatives.

In this course we will basically generalize all these notions and techniques to functions which depend on more than one real variable.

In the last part of this course we will come back to functions of one variable and learn how to solve specific types of differential equations, a little more complicated than those you studied last year. In particular, we will learn some methods to solve equations of the generic type

$$\ddot{y}(x) + a\dot{y}(x) + by = R(x), \tag{1.5}$$

where a, b are constants and $R(x)$ is a real function of x. These are so-called linear second-order differential equations with constant coefficients.

2 Functions of several real variables

In order to familiarize yourself with the idea of functions of two real variables and what they look like, you can have a look at the following web-sites.

- http://www.ucl.ac.uk/Mathematics/geomath/level2/pdiff/pd5.html
  http://tutorial.math.lamar.edu/Classes/CalcIII/MultiVrbleFcns.aspx
  These web-sites show you some pictures of functions of two variables and explains how to draw them.

- http://www-math.mit.edu/18.013A/HTML/tools/tools22.html
  This web-site allows you to enter the equation of a function of two variables, plot it and look at it from different directions.

- http://www.plot3d.net/index.html
  From this web-site you can download the software plot3d that allows you to plot surfaces (functions of two variables).

You will be able to find lots of other interesting links in the web!

2.1 Basic definitions and examples

Definition: A function f of n variables is a rule that assigns a **unique** real number $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$ to each point $(x_1, x_2, \ldots, x_n)$ contained in some subset $\mathcal{D}(f)$ of $\mathbb{R}^n$. The set of points $\mathcal{D}(f)$ is called the **domain** of the function f. The set of all real numbers $f(x_1, x_2, \ldots, x_n)$ obtained from points in the domain is called the **range** of f and we will denote it by $\mathcal{R}(f)$. Another way to say the same is

$$\begin{aligned} f : \mathcal{D}(f) \subset \mathbb{R}^n &\rightarrow \mathcal{R}(f) \subset \mathbb{R} \\ (x_1, x_2, \ldots, x_n) \in \mathcal{D}(f) &\rightarrow f(x_1, x_2, \ldots, x_n) \in \mathcal{R}(f) \end{aligned} \tag{2.1}$$

As we said before, in this course we will basically study functions of 2 variables, which are special cases of the general definition above:

Definition: A function f of two variables is a rule that assigns a **unique** real number $f(x, y) \in \mathbb{R}$ to each point (x, y) contained in some subset $\mathcal{D}(f)$ of $\mathbb{R}^2$. The set of points $\mathcal{D}(f)$ is called the **domain** of the function f. The set of all real numbers $f(x, y)$ obtained from points in the domain is called the **range** of f and we will denote it by $\mathcal{R}(f)$

$$\begin{aligned} f : \mathcal{D}(f) \subset \mathbb{R}^2 &\rightarrow \mathcal{R}(f) \subset \mathbb{R} \\ (x, y) \in \mathcal{D}(f) &\rightarrow f(x, y) \in \mathcal{R}(f) \end{aligned} \tag{2.2}$$

Important notation: You might be used to employ the notation $P(x, y)$ to denote a point in the 2D-plane. Here we will instead use the notation (x, y). Analogously we will denote by (x, y, z) a point in the 3-dimensional space. In general, $(x_1, x_2, \ldots, x_n)$ will denote a point in a n-dimensional space. We denote by $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^n$ the 2D-plane, the 3D-space and the n-dimensional space, respectively.

For functions of one variable you have already seen that sometimes we call $y = f(x)$, since the value of the function is represented in the y axis of the xy-plane. Similarly, we will often call

$z = f(x, y)$ for functions of two variables, since the values of the functions will be represented in the z-axis (see picture below).
All the definitions above are illustrated in the following figure for a generic function of 2 variables

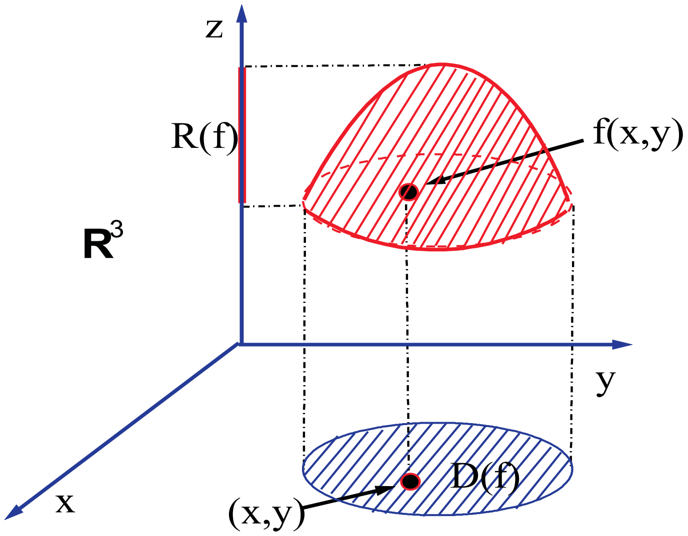


Figure 1: *Graphic representation of a generic function of two real variables $f(x, y)$. The function is a surface which is represented in red and that has more or less the shape of a hat. The range and domain of the function are also shown.*

Figure 1 shows also the standard choice for the orientation of the axes that we will use. This is what we call a right-handed coordinate system. The values of the the function $f(x, y)$ are represented in the z axis. In the picture you can also see the 2-dimensional region which is the domain $\mathcal{D}(f)$ (in blue) as well as the set of points in the z axis which gives the range of values of f, $\mathcal{R}(f)$ (as a red line).

In general, the domain of a function of two variables can either be the whole 2-dimensional xy-plane (that is $\mathbb{R}^2$) or just some region of it (as in figure 1). An alternative way of defining $\mathcal{D}(f)$ is to say that it is the set of points (x, y) at which the function $f(x, y)$ exists.

<u>How to determine the domain of a function of two variables?</u> As a general rule, if a function $f(x, y)$ is given and no information about its domain is provided we must conclude that its domain is the whole xy-plane except for those points (if any) at which the function does not make sense. There are two main situations in which a function f does not make sense at a certain point (x_0, y_0): if the value $f(x_0, y_0)$ is not a real number or if the function $f(x_0, y_0)$ diverges at (x, y). Let us study some examples:

Example 1: Consider the function

$$z = f_1(x, y) = x^2 + xy. \tag{2.3}$$

This function is well-defined for all values of x and y. Therefore, since nothing is said about its domain, we conclude that the domain are all points $(x, y) \in \mathbb{R}^2$. The function is well-defined because it takes real values for any real values of (x, y) and it does not become infinity for any finite values of x and y.

Example 2: Let

$$z = f_2(x,y) = 3\left(1 - \frac{x}{2} - \frac{y}{4}\right), \qquad \text{for} \qquad 0 \le x \le 2 \quad \text{and} \quad 0 \le y \le 4-2x. \tag{2.4}$$

In this case a specific domain is given, therefore even though the function is well defined for all real values of x and y, its domain is the one shown. The domain above tells us that $x \in [0,2]$ and that for $x = 0$, y can vary in the range $0 \le y \le 4$, whereas for $x = 2$, y can only be 0. With this information we can sketch the domain of $f_2(x,y)$, which is nothing but a triangle in the (x,y) plane, with vertices at the points $(0,0), (2,0)$ and $(0,4)$ (see figure 2 below).

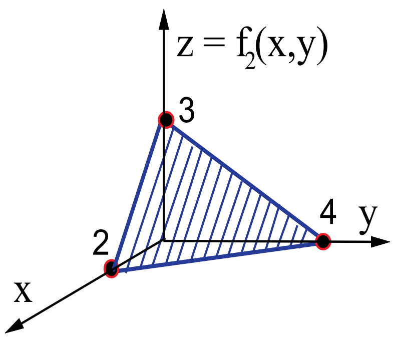
Example 3: Let

$$z = f_3(x,y) = \sqrt{9 - x^2 - y^2}. \tag{2.5}$$

This function is only well defined when the expression under the square root is not negative. Therefore, its domain is defined as the set of points (x,y) satisfying the condition

$$x^2 + y^2 \le 9. \tag{2.6}$$

This condition defines a disk of radius 3 in the xy-plane*. A plot of the function $f_3(x,y)$ in the xyz-space generates a half-sphere in the upper half plane for $z \ge 0$ (see figure 2 below).



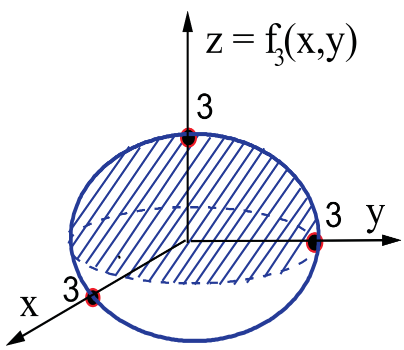


Figure 2: *Graphic representation of the functions (2.4) and (2.5). The dashed regions are the 2D-surfaces generated by the functions, which in this case are a triangle and a half-sphere, respectively.*

Example 4: Finally, let us consider the function

$$z = f_4(x,y) = \frac{1}{x-y}. \tag{2.7}$$

Here, no domain is specified, so in principle it should be all the $\mathbb{R}^2$ plane. However we see that the function is not well-defined whenever $x = y$. Therefore the domain of f_4 is the whole xy-plane minus the line $x = y$.

*This is so since $\sqrt{x^2+y^2}$ gives the distance of a point (x,y) to the origin and therefore the points (2.6) are just the set of points (x,y) whose distance to the origin is 3.

Definition: A function $f(x,y)$ is said to be **bounded** if its range $\mathcal{R}(f)$ lies within a finite interval.

Example: Consider again the function $f_3(x,y)$

$$z = f_3(x,y) = \sqrt{9 - x^2 - y^2}, \tag{2.8}$$

whose domain was discussed above. It is clear that the maximum value the function can take is 3, corresponding to $x = y = 0$, whereas the minimum would be 0 (as the function can never be negative). Therefore this is a bounded function with range

$$\mathcal{R}(f_3) = \{z \in \mathbb{R} \,|\, 0 \leq z \leq 3\}.$$

2.2 Limits and continuity

In order to introduce the concepts of limit and continuity for functions of more than one variable we need first to generalise the concept of **neighbourhood of a point** from $\mathbb{R}$ to $\mathbb{R}^2$ and $\mathbb{R}^3$. Let us start by recalling the corresponding definition for functions of one variable.

2.2.1 Neighbourhood of a point in one, two and three dimensions

Definition: If x_0 is a point in the real line and $\varepsilon > 0$, then the set of points x in the real line whose distances from x_0 are less than ε is called an **open interval about** x_0 or a **neighbourhood of** x_0. If we denote this neighbourhood by $\mathcal{N}_\varepsilon(x_0)$, then

$$\mathcal{N}_\varepsilon(x_0) = \{x : |x - x_0| < \varepsilon\} \tag{2.9}$$

Definition: If (x_0, y_0) is a point in the 2-dimensional xy-plane and $\varepsilon > 0$, then the set of points (x, y) in the plane whose distances from (x_0, y_0) are less than ε is called an **open disk of radius** ε **about** (x_0, y_0) or, more simply, a **neighbourhood of** (x_0, y_0). If we denote this neighbourhood by $\mathcal{N}_\varepsilon((x_0, y_0))$, then

$$\mathcal{N}_\varepsilon((x_0, y_0)) = \left\{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \varepsilon\right\}. \tag{2.10}$$

Note: Since the equation

$$(x - x_0)^2 + (y - y_0)^2 = \varepsilon^2, \tag{2.11}$$

defines a circle of radius ε centered at the point (x_0, y_0), the neighbourhood of (x_0, y_0) are all the points contained inside that circle (that is, a disk) but **not** the points on the circle.

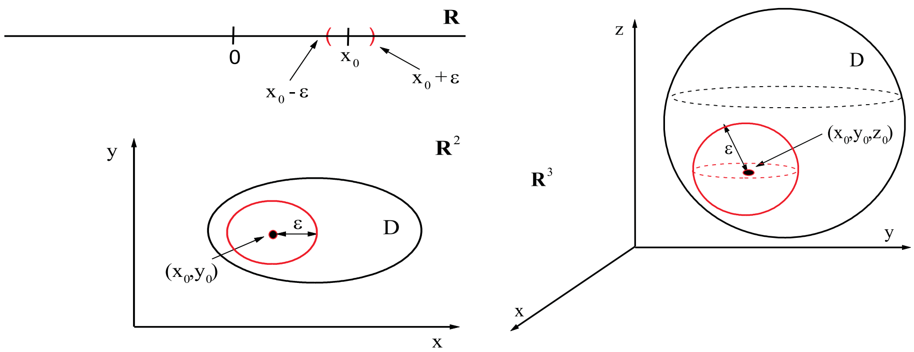


Figure 3: *Neighbourhood of a point in one, two and three dimensions. The regions D are arbitrary subspaces of* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *where the point and its neighbourhood live.*

Definition: If (x_0, y_0, z_0) is a point in the 3-dimensional xyz-plane and $\varepsilon > 0$, then the set of points (x, y, z) in the plane whose distances from (x_0, y_0, z_0) are less than ε is called an **open ball of radius** ε **about** (x_0, y_0, z_0) or, more simply, a **neighbourhood of** (x_0, y_0, z_0). If we denote this neighbourhood by $\mathcal{N}_\varepsilon((x_0, y_0, z_0))$, then

$$\mathcal{N}_\varepsilon((x_0, y_0, z_0)) = \left\{(x, y, z) : \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \varepsilon\right\}. \tag{2.12}$$

Note: Since the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \varepsilon^2, \tag{2.13}$$

defines a sphere of radius ε centered at the point (x_0, y_0, z_0), the neighbourhood of (x_0, y_0, z_0) are all the points contained inside that sphere (that is, a ball) but **not** the points lying on the sphere.

It is convenient here to introduce some notions related to sets in the 2-dimensional space:

Definition: Let S be a set of points in the plane:

- We say that a point (x_0, y_0) is a **boundary point** of S if every neighbourhood of (x_0, y_0) contains at least one point in S and at least one point not in S.
- We say that S is closed (or is a **closed set**) if every boundary point of S belongs to S.
- We say that S is open (or is an **open set**) if no boundary point of S belongs to S.
- The **interior** of S is the set of all points in S that are not boundary points of S. The **exterior** of S is the set of all points that are not in S and not boundary points of S.

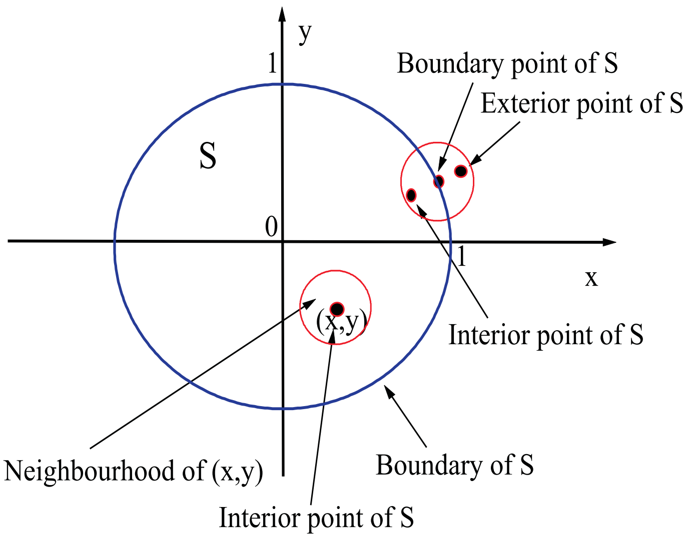


Figure 3: *Boundary points, interior and exterior points corresponding to the set S defined by $x^2+y^2 \leq 1$, that is the closed disk of radius 1.*

2.2.2 Limits of functions of two variables

The concept of limit of a function of several variables is similar to that for functions of one variable. For clarity we will present here only the case of functions of 2 variables; the general case is a natural generalization of that one.

Definition: We say that a function $f(x,y)$ approaches the limit L as the point (x,y) approaches the point (x_0,y_0) and we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L, \tag{2.14}$$

if all points of a neighbourhood of (x_0,y_0), except possibly the point (x_0,y_0) itself, belong to the domain $\mathcal{D}(f)$ of f, and if $f(x,y)$ approaches L as (x,y) approaches (x_0,y_0). In other words, the closer the point (x,y) is to (x_0,y_0), the closer is the function $f(x,y)$ to the limiting value L.

A more formal definition of limit: We say that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L, \tag{2.15}$$

- every neighbourhood of (x_0,y_0) contains points (x,y) in $\mathcal{D}(f)$, and
- if and only if for every number $\varepsilon > 0$ there exists another number $\delta(\varepsilon) > 0$ such that

$$\text{if} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta(\varepsilon) \quad \text{then} \quad |f(x,y) - L| < \varepsilon, \tag{2.16}$$

  whenever $(x,y) \in \mathcal{D}(f)$.

Laws of limits: All usual laws of limits extend to functions of several variables in the obvious way. For example if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M, \tag{2.17}$$

then

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm M \tag{2.18}$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)g(x,y) = LM, \tag{2.19}$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}. \tag{2.20}$$

Existence of the limit:

- If a limit exists it is unique.
- For a function of one variable $f(x)$ the existence of the limit $\lim_{x\to x_0} f(x)$ implies that the function $f(x)$ approaches the same finite number as x approaches x_0 from either the right or the left.
- For a function of 2 variables $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ exists only if $f(x,y)$ approaches the same number L **no matter how** (x,y) approaches (x_0,y_0) in the xy-plane. That means that (x,y) can approach (x_0,y_0) along any curve in the xy-plane which goes through (x_0,y_0).
- Notice that it is not necessary that $f(x_0,y_0) = L$, even if $f(x_0,y_0)$ is defined.

Let us now exemplify these definitions with various examples:

Example 1: Let us consider the following limits

$$\lim_{(x,y)\to(1,1)} 2x - y^2 = 2 - 1 = 1, \tag{2.21}$$

$$\lim_{(x,y)\to(x_0,y_0)} x^2 y = x_0^2 y_0, \tag{2.22}$$

$$\lim_{(x,y)\to(\pi,2)} y\sin(xy) = 2\sin(2\pi) = 0. \tag{2.23}$$

All these examples show limits which exist, in the sense that the functions approach the same limit no matter how (x, y) approaches the limiting point. In fact, in all these cases the limit of the function coincides with the value of the function at the limiting point. As we said above this is not always the case.

Example 2: Let

$$f_2(x,y) = \frac{2xy}{x^2+y^2}, \tag{2.24}$$

and try to compute the limit

$$\lim_{(x,y)\to(0,0)} f_2(x,y). \tag{2.25}$$

It is not difficult to prove that this limit **does not exist**. This is due to the fact that the function $f(x, y)$ approaches different limiting values depending on the way the point (x, y) approaches $(0, 0)$. To see that, suppose that we take the limit $(x, y) \to (0, 0)$ along curves parameterized by $y = kx$, where k is an arbitrary real number different from 0,

$$\lim_{(x,kx)\to(0,0)} f_2(x,kx) = \lim_{(x,kx)\to(0,0)} \frac{2kx^2}{x^2+k^2x^2} = \frac{2k}{1+k^2}. \tag{2.26}$$

Obviously, the value of the limit is different for each different value of k. Therefore the limit is not unique which implies it does not exist. This is also an example of a function which is not well defined at the limiting point $(0, 0)$, since $f(0, 0) = 0/0 =$ indeterminate.

Example 3: Let

$$f_3(x,y) = \frac{2x^2y}{x^4+y^2}. \tag{2.27}$$

We want to compute the limit

$$\lim_{(x,y)\to(0,0)} f_3(x,y). \tag{2.28}$$

We can try to compute the limit along lines parameterized as before, $y = kx$ with $k \neq 0$. We obtain

$$\lim_{(x,kx)\to(0,0)} f_3(x,kx) = \lim_{x\to 0} \frac{2kx^3}{x^4+k^2x^2} = \lim_{x\to 0} \frac{2kx}{x^2+k^2} = 0. \tag{2.29}$$

So the limit along any straight line $y = kx$ is 0. We might be then tempted to conclude that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, but this would be incorrect. To see that let us now take the limit along curves of the form $y = kx^2$. Then

$$\lim_{(x,kx^3)\to(0,0)} f_3(x,kx^2) = \lim_{x\to 0} \frac{2kx^4}{x^4+k^2x^4} = \frac{2k}{1+k^2}. \tag{2.30}$$

So, the limit along curves $y = kx^2$ is not 0 but gives a different value for each different choice of k. Therefore the limit (2.28) does not exist.

Example 4: Let

$$f_4(x, y) = \frac{x^2 y}{x^2 + y^2}. \tag{2.31}$$

and prove that the limit

$$\lim_{(x,y)\to(0,0)} f_4(x, y) = 0. \tag{2.32}$$

We can try to prove that by choosing some particular curves along which to take the limit, as in the previous examples. However, to be really sure that the limit is always the same, no matter which curve we take, we can try to prove (2.32) from the rigorous definition of the limit given in (2.16). Since $x^2 \leq x^2 + y^2$ and $y^2 \leq x^2 + y^2$, we have that

$$|f_4(x, y) - 0| = \left| \frac{x^2 y}{x^2 + y^2} \right| \leq |y| \leq \sqrt{x^2 + y^2}. \tag{2.33}$$

So, formally if we consider a number $\varepsilon > 0$ and we take $\delta(\varepsilon) = \varepsilon$ it is guaranteed that if

$$0 < \sqrt{x^2 + y^2} < \varepsilon, \tag{2.34}$$

then

$$|f_4(x, y) - 0| < \varepsilon, \tag{2.35}$$

therefore the limit according to definition (2.16) exists and is (2.32).

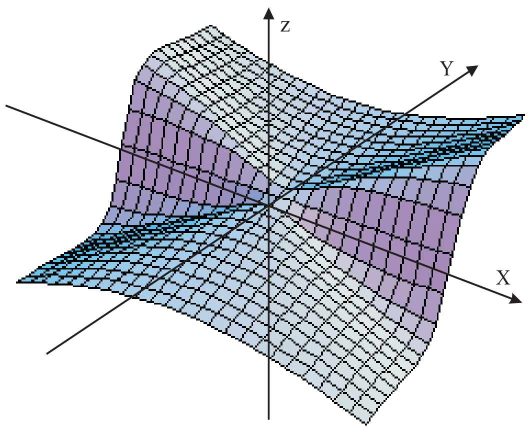


Figure 4: *Graphic representation of the function $f_4(x, y) = x^2 y/(x^2 + y^2)$.*

2.2.3 Continuity of functions of two variables

Definition: The function $f(x, y)$ is **continuous** at the point (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0). \tag{2.36}$$

<u>Note:</u> Notice that it **is not always true** that the limit of a function at a certain point coincides with the value of the function at that point. Therefore the condition (2.36) is a non-trivial constraint. For example, we can define the following function

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases} \tag{2.37}$$

It is clear that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0, \tag{2.38}$$

since when performing the limit we approach the point $(0,0)$ "infinitely" but we never exactly reach it. Therefore the value of the limit is in this case the value of the function everywhere but at $(0,0)$, which is by definition 0. However $f(0,0) = 1$. Therefore, this function is discontinuous at $(0,0)$. We can easily make this function continuous everywhere by simply changing its value at the origin to $f(0,0) = 0$.

2.3 Differentiation of functions of several real variables

In this section we will begin the process of extending the concepts and techniques of single-variable calculus to functions of more than one variable. It is convenient to begin by considering the rate of change of such functions with respect to one variable at a time. This is what we will call **first-order partial derivative**. We will denote the first-order partial derivative with respect to the variable x as

$$\frac{\partial}{\partial x}, \tag{2.39}$$

in order to distinguish it from the total derivative

$$\frac{d}{dx}, \tag{2.40}$$

which we used for functions of one variable. Thus, a function of n variables has n first-order partial derivatives, one with respect to each of its independent variables

$$\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_i, \ldots, x_n) \quad \text{for} \quad i = 1, \ldots, n. \tag{2.41}$$

Let us now provide a more rigorous definition for functions of just two variables.

2.3.1 Definition of partial derivative for a function of two variables

Definition: The first partial derivatives of the function $f(x, y)$ with respect to the variables x and y are given by

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \tag{2.42}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}. \tag{2.43}$$

Note: Notice that $\partial f/\partial x$ is nothing but the standard first derivative of $f(x, y)$ when considered as a function of x only, regarding y as a constant parameter. Similarly $\partial f/\partial y$ is the standard first derivative of $f(x, y)$ when considered as a function of y only, regarding x as a constant parameter.

Example: Let

$$f(x, y) = x^2 \cos y + 2xy, \tag{2.44}$$

then, according to the definition above, the partial derivatives of $f(x, y)$ with respect to x and y are

$$\frac{\partial f}{\partial x} = \cos y \frac{dx^2}{dx} + 2y \frac{dx}{dx} = 2x \cos y + 2y, \tag{2.45}$$

$$\frac{\partial f}{\partial y} = x^2 \frac{d \cos y}{dy} + 2x \frac{dy}{dy} = -x^2 \sin y + 2x. \tag{2.46}$$

Geometric interpretation: Partial derivatives of functions of two variables admit a similar geometrical interpretation as for functions of one variable. For a function of one variable $f(x)$, the first derivative with respect to x is defined as

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{2.47}$$

and geometrically it measures the slope of the curve $f(x)$ at the point x. This is illustrated in figure 5.

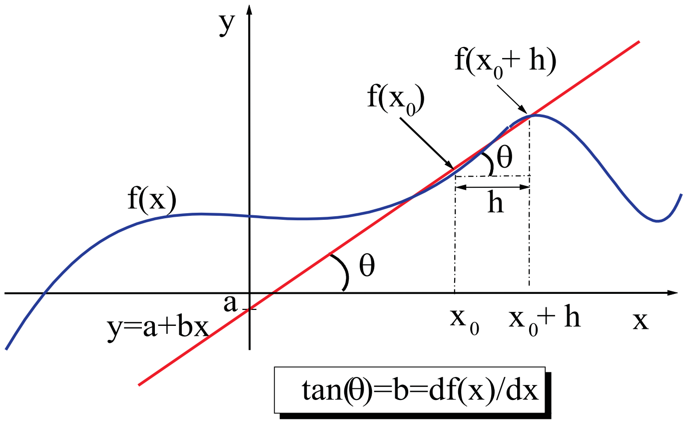


Figure 5: *Geometrical picture of df/dx at the point $x = x_0$. Applying the definition (2.47), the derivative is nothing but the slope of the line $y = a + bx$ which is tangent to $f(x)$ at the point x_0, namely $b = \frac{df}{dx}|_{x_0}$.*

Recalling the definition we gave in the note above, we can now interpret geometrically the first-order partial derivatives of a function of two variables in a completely analog fashion:

<u>**Geometrical definition of f_x and f_y:**</u> The partial derivative $\partial f/\partial x$ at a certain point (x_0, y_0) is nothing but the slope of the curve of intersection of the function $f(x, y)$ and the vertical plane $y = y_0$ at $x = x_0$. Likewise, the partial derivative $\partial f/\partial y$ at a certain point (x_0, y_0) is nothing but the slope of the curve of intersection of the function $f(x, y)$ and the horizontal plane $x = x_0$ at $y = y_0$. Graphically:

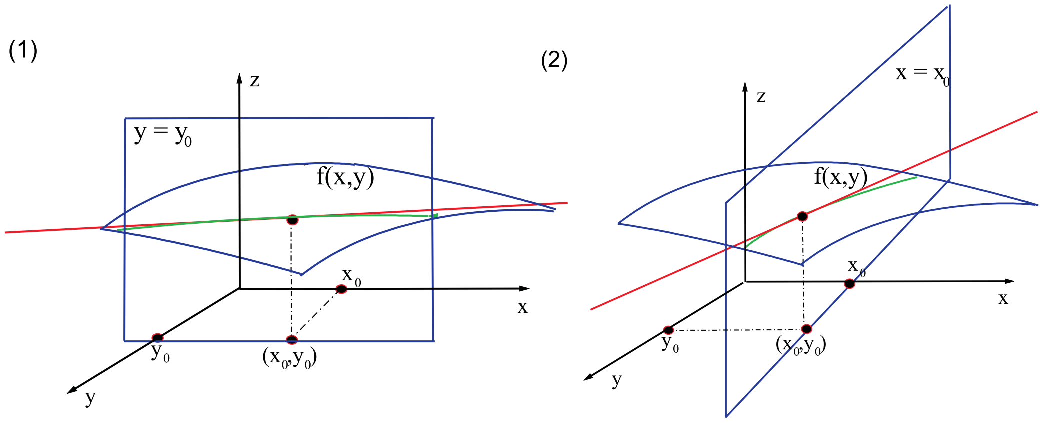


Figure 6: *Geometrical picture of $\partial f/\partial x$ (1) and $\partial f/\partial y$ (2) at the point (x_0, y_0). In (1) $\partial f/\partial x$ is the slope of the red line which is tangent to the green curve resulting from the intersection of $f(x, y)$ and the plane $y = y_0$. In (2) $\partial f/\partial y$ can be identified in a similar fashion.*

The geometric interpretation of partial derivatives is also rather well explained in:
http://www.math.uri.edu/Center/workht/calc3/tangent1.html
and can be visualized in
http://www-math.mit.edu/18.013A/HTML/tools/tools22.html

<u>**Notation:**</u> From now on we will employ the following shorter notation for the partial derivatives of $f(x, y)$

$$\frac{\partial f}{\partial x} = f_x, \qquad \frac{\partial f}{\partial y} = f_y. \tag{2.48}$$

We will denote by $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ the partial derivatives at the point (x_0, y_0).

Definition: Given a function $f(x, y)$, the function is said to be **differentiable** if f_x and f_y exist. If the function is differentiable its first-order derivatives can be differentiated again and we can define the **second-order partial derivatives** of $f(x, y)$ as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \tag{2.49}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \tag{2.50}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}, \tag{2.51}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}. \tag{2.52}$$

Note: Notice that f_{xy} and f_{yx} are in principle different functions. Using the definitions (2.42)-(2.43) twice we can see that the second derivatives f_{xy} and f_{yx} are given by the double limits

$$\begin{aligned} f_{xy} &= \lim_{p\to 0} \frac{f_y(x+p, y) - f_y(x, y)}{p} \qquad (2.53)\\ &= \lim_{p\to 0}\left[\lim_{h\to 0} \frac{f(x+p, y+h) - f(x+p, y) - f(x, y+h) + f(x, y)}{hp}\right], \\ f_{yx} &= \lim_{h\to 0} \frac{f_x(x, y+h) - f_x(x, y)}{h} \qquad (2.54)\\ &= \lim_{h\to 0}\left[\lim_{p\to 0} \frac{f(x+p, y+h) - f(x+p, y) - f(x, y+h) + f(x, y)}{hp}\right]. \end{aligned}$$

Therefore, the only difference between f_{xy} and f_{yx} is the order in which the limits are taken. **It is not guaranteed that the limits commute**.

Example 1: Let us compute the first- and second-order partial derivatives of the function

$$f(x, y) = e^{xy} + \ln(x^2 + y). \tag{2.55}$$

We start with the 1st derivatives,

$$f_x = ye^{xy} + \frac{2x}{x^2 + y}, \tag{2.56}$$

$$f_y = xe^{xy} + \frac{1}{x^2 + y}. \tag{2.57}$$

Now we can compute the 2nd derivatives,

$$f_{xy} = \frac{\partial f_y}{\partial x} = e^{xy} + xye^{xy} - \frac{2x}{(x^2 + y)^2}, \tag{2.58}$$

$$f_{yx} = \frac{\partial f_x}{\partial y} = e^{xy} + xye^{xy} - \frac{2x}{(x^2 + y)^2}. \tag{2.59}$$

So, in this case $f_{xy} = f_{yx}$.

Example 2: As we said at the beginning of this section, all definitions for functions of two variables extend easily to functions of 3 or more variables. In this example let us consider the function of three variables

$$g(x, y, z) = e^{x-2y+3z}, \tag{2.60}$$

and compute its 1st and 2nd order partial derivatives. In this case we have 3 1st order derivatives

$$g_x = e^{x-2y+3z}, \qquad g_y = -2e^{x-2y+3z}, \qquad g_z = 3e^{x-2y+3z}. \tag{2.61}$$

Now we have in total 9 possible different 2nd order derivatives,

$$g_{xx} = e^{x-2y+3z}, \qquad g_{yx} = -2e^{x-2y+3z}, \qquad g_{zx} = 3e^{x-2y+3z}, \tag{2.62}$$
$$g_{xy} = -2e^{x-2y+3z}, \qquad g_{yy} = 4e^{x-2y+3z}, \qquad g_{zy} = -6e^{x-2y+3z}, \tag{2.63}$$
$$g_{xz} = 3e^{x-2y+3z}, \qquad g_{yz} = -6e^{x-2y+3z}, \qquad g_{zz} = 9e^{x-2y+3z}. \tag{2.64}$$

Definition: Consider a function of two variables $f(x, y)$ and let f,f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx} exist and be **continuous** in a neighbourhood of a point (x_0, y_0). Then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0). \tag{2.65}$$

A nice proof of this theorem is given in chapter 13 of the book "Calculus" by R. Adams. The key idea is that the continuity of all 1st and 2nd order derivatives of f allows us to proof that the order of the limits in (2.53) and (2.54) is irrelevant for the final result.

2.3.2 Chain rules

Definition: The chain rule for functions of one variable is a formula that gives the derivative of the composition of two functions f and g, that is the derivative of the function $f(x)$ with respect to a new variable t, df/dt for $x = g(t)$. We know from last year's Calculus that,

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = f'(x)g'(t), \tag{2.66}$$

On the other hand, if we use the geometric definition of the derivative seen above we can also write

$$\frac{df}{dt} = \lim_{h\to 0}\frac{f(g(t+h)) - f(g(t))}{h}. \tag{2.67}$$

In other words, (2.66) and (2.67) must be equal. We will use this identity later on for functions of two variables.

Example: Let $f(x) = 1/x + x^2$ and $x = t^2 + t + 1$, then

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = \left(-\frac{1}{x^2} + 2x\right)(2t+1) = \left(-\frac{1}{(t^2+t+1)^2} + 2(t^2+t+1)\right)(2t+1).$$

Definition: The chain rule for functions of two variables becomes considerably more complicated than for functions of one variable, but the principle is the same. Suppose we have a function of two variables, $f(x, y)$ and we consider the change of variables $x = u(t)$ and $y = v(t)$. The question is, how do we compute the derivative df/dt in terms of the partial derivatives f_x and f_y? If we use the definition of the derivative by means of a limit we know that

$$\begin{aligned}
\frac{df}{dt} &= \lim_{h\to 0}\frac{f(u(t+h), v(t+h)) - f(u(t), v(t))}{h} \\
&= \lim_{h\to 0}\underbrace{\frac{f(u(t+h), v(t+h)) - {\color{blue}f(u(t), v(t+h))}}{h}}_{\text{increment with } v(t+h) \text{ fixed}} \\
&+ \lim_{h\to 0}\underbrace{\frac{{\color{blue}f(u(t), v(t+h))} - f(u(t), v(t))}{h}}_{\text{increment with } u(t) \text{ fixed}} \\
&= \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} \equiv f_x u_t + f_y v_t.
\end{aligned} \tag{2.68}$$

In the 2th and 3th line of (2.68) we have managed to separate the 1st line into the sum of two quotients by adding zero to the 1st line in an smart way (notice that the bits in blue cancel each other). The first of these quotients (2th line) involves changes only on the first variable $u(t)$ whereas in the second quotient (3th line) we have changes only in the second variable $v(t)$. Notice that above we have used $\partial f/\partial u = f_x$ and $\partial f/\partial v = f_y$, simply because $x = u(t)$ and $y = v(t)$. Now we can use the chain rule for functions of one variable (2.67) to obtain the final expression in the 4th line.

Generalizations of the chain rule: The formula above can be also generalized to the case when a more general change of variables in considered, namely:

$$x = u(s, t), \qquad y = v(s, t), \tag{2.69}$$

that is, when the variables x, y are functions of two other variables s and t. In this case the chain rule tells us that the partial derivatives f_s and f_t can be obtained as

$$f_s = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s} \equiv f_x u_s + f_y v_s, \tag{2.70}$$

$$f_t = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} \equiv f_x u_t + f_y v_t, \tag{2.71}$$

provided that f_x and f_y are continuous functions. Notice that these equations can be written in matrix form

$$(g_s, g_t) = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right) \cdot \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix}. \tag{2.72}$$

This matrix form is very convenient for generalizations to functions of more than 2 variables. For example, given a function of n variables $f(x_1, \ldots x_n)$ such that

$$x_i = u_i(s_1, \ldots, s_m) \quad \forall \quad i = 1, \ldots, n, \tag{2.73}$$

the chain rule can be written as

$$\left(\frac{\partial f}{\partial s_1}, \ldots, \frac{\partial f}{\partial s_m}\right) = \left(\frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_n}\right) \cdot \begin{pmatrix} \frac{\partial u_1}{\partial s_1} & \cdots & \frac{\partial u_1}{\partial s_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial s_1} & \cdots & \frac{\partial u_n}{\partial s_m} \end{pmatrix}, \tag{2.74}$$

or equivalently

$$\frac{\partial f}{\partial s_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\frac{\partial u_i}{\partial s_j} \quad \forall \quad i = 1, \ldots, n, \quad \text{and} \quad j = 1, \ldots, m. \tag{2.75}$$

provided that the 1st order partial derivatives $\partial f/\partial u_i$ are continuous. The matrix in (2.74) is called the **Jacobian matrix of the variable transformation**.

Let us consider now several examples:

Example 1: Consider the function

$$f(x, y) = \sin(x + y) \quad \text{with} \quad x = st^2 \quad \text{and} \quad y = s^2 + 1/t. \tag{2.76}$$

Compute f_s and f_t in two possible ways.
Two possible ways in which we can compute these partial derivatives are

- by using the chain rule,
- or by replacing x and y in $f(x, y)$ by their expressions in terms of s and t and then computing f_s and f_t directly.

If we use the chain rule we will need the following partial derivatives:

$$f_x = \cos(x + y), \qquad f_y = \cos(x + y), \tag{2.77}$$

$$x_s = t^2, \qquad x_t = 2ts, \tag{2.78}$$

$$y_s = 2s, \qquad y_t = -1/t^2. \tag{2.79}$$

Then f_s and f_t are simply given by

$$\begin{aligned}
f_s &= f_x x_s + f_y y_s = \cos(x+y)(2s+t^2) \\
&= \cos(st^2+s^2+\frac{1}{t})(2s+t^2), && (2.80) \\
f_t &= f_x x_t + f_y y_t = \cos(x+y)(2ts-\frac{1}{t^2}) \\
&= \cos(st^2+s^2+\frac{1}{t})(2ts-\frac{1}{t^2}). && (2.81)
\end{aligned}$$

The second way to compute these derivatives is to substitute x and y in terms of s and t in $f(x,y)$. By doing that we obtain

$$f(x(s,t),y(s,t)) = \sin(st^2+s^2+1/t). \tag{2.82}$$

Now we can obtain f_s and f_t directly as

$$\begin{aligned}
f_s &= (t^2+2s)\cos(st^2+s^2+1/t), && (2.83) \\
f_t &= (2st-1/t^2)\cos(st^2+s^2+1/t). && (2.84)
\end{aligned}$$

Notice that here we have called $f(x,y)$ and $f(x(s,t),y(s,t))$ both f (before, we have been using different names). We will keep doing this in the future, as it makes things simpler.

Example 2: Consider now a function of three variables $f(x,y,z)$ with $x=g(z)$ and $y=h(z)$. How can we compute the derivative df/dz?

We can again apply the chain rule now for a function of three variables which in this case are all functions of the same variable z. We need only to use our general formula (2.75) with $n=3$ and $m=1$, that is

$$\frac{df}{dz} = \frac{\partial f}{\partial x}\frac{dx}{dz} + \frac{\partial f}{\partial y}\frac{dy}{dz} + \frac{\partial f}{\partial z}. \tag{2.85}$$

Example 3: Suppose that the temperature T of a certain liquid varies with the depth of the liquid z and the time t as $T(z,t)=e^{-t}z$. What is the rate of change of the temperature with respect to the time at a point that is moving through the liquid in such a way that at time t its depth is $z=f(t)$? What is this rate if $f(t)=e^t$? What is happening in this case?

Here we have an example of a function of two variables $T(z,t)$ and they tell us to compute $\partial T/\partial t$ for a point such that $z=f(t)$. This is a clear case when we can use the chain rule

$$\frac{dT}{dt} = \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t} = e^{-t}f'(t) - ze^{-t} = e^{-t}f'(t) - f(t)e^{-t}. \tag{2.86}$$

If in particular $f(t)=e^t$, then the previous formula gives

$$\frac{dT}{dt} = e^{-t}f'(t) - f(t)e^{-t} = 1-1 = 0. \tag{2.87}$$

In this case the decrease in temperature due to the increase of depth and the decrease in temperature due to the increase of time are perfectly balanced in such a way that the temperature does not change with time.

2.3.3 Definition of differential

As in previous sections, it is useful to start this section by recalling the definition of differential for functions of one variable:

Definition: Given a function $f(x)$ and assuming that its total derivative df/dx exists at a certain point x, the total differential df of the function is given by

$$df = \left(\frac{df}{dx}\right) dx = f'(x)dx. \tag{2.88}$$

The quantity df can be interpreted as the infinitesimal change on the value of the function $f(x)$ when x changes by the infinitesimal amount dx. A mathematical way of expressing this is to define

$$\Delta f = f(x + \Delta x) - f(x), \tag{2.89}$$

where Δf and Δx are finite increments of f and x. Then one needs to prove that

$$\Delta f \to df \qquad \text{as} \qquad \Delta x \to dx.$$

To prove this we can use the mean value theorem† which allows us to rewrite (2.89) as

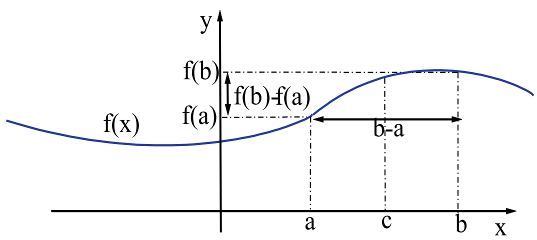
$$\Delta f = f'(x + \theta\Delta x)\Delta x \quad \text{with} \quad \theta \in (0, 1). \tag{2.93}$$

Now we can take the limit when $\Delta x \to dx$ which implies $\Delta f \to df$ and dx, df being infinitesimal increments. Since dx is very small we can use $dx \ll x$, to obtain

$$df = f'(x)dx. \tag{2.94}$$

which is nothing but (2.88).

†Given a function $f(x)$ which is continuous and has continuous 1st total derivative $f'(x)$ the mean value theorem tells us that if a, b are points at which the function takes values $f(a)$ and $f(b)$ and $a < b$, then a third point c exists, $c \in (a, b)$ such that



$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{2.90}$$

If we define

$$\theta = \frac{c - a}{b - a}, \tag{2.91}$$

we have $\theta \in (0, 1)$ and we can rewrite (2.90) as

$$f(b) - f(a) = f'(a + \theta(b - a))(b - a). \tag{2.92}$$

Definition: A similar quantity can be defined for functions of more than one variable. For example, let us consider now a function of two variables $f(x, y)$ with continuous first order partial derivatives f_x and f_y. We define the total differential of f,

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy = f_x dx + f_y dy, \tag{2.95}$$

as the small variation experienced by f when the variables x and y are changed by infinitesimal amounts dx and dy respectively. As before we can define

$$\begin{aligned} \Delta f &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= \underbrace{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}_{\text{increment with } y+\Delta y \text{ fixed}} + \underbrace{f(x, y + \Delta y) - f(x, y)}_{\text{increment with } x \text{ fixed}}, \end{aligned} \tag{2.96}$$

now we have managed to split Δf into two pieces, each of which involves a variation only in x and only in y, respectively. These two terms are now analogous to the definition (2.89) for a function of one variable and this allows us to write

$$\Delta f = f_x(x + \theta_1 \Delta x, y + \Delta y)\Delta x + f_y(x, y + \theta_2 \Delta y)\Delta y, \quad \text{with} \quad \theta_1, \theta_2 \in (0, 1). \tag{2.97}$$

Now, as for functions of one variable, in the limit $\Delta x \to dx$ and $\Delta y \to dy$ with dx, dy being infinitesimal increments ($dx \ll x$ and $dy \ll y$), then $\Delta f \to df$ and we recover the result (2.95).

The differential is a very useful concept when we want to obtain approximate values of a function nearby a point at which the value of the function and its partial derivatives are known. A good example of this is example 2 below. Example 1 is a practical example of how to compute the differential of a function of two variables:

Example 1: The fundamental equation which characterizes an ideal gas is

$$RT = PV, \tag{2.98}$$

where R is a universal constant and P, V, T are the three state variables (pressure, volume and temperature of the gas). Obtain the change in the pressure of the gas due to a small change of its volume and temperature.

This is a typical exercise in which they ask us to compute the differential of the pressure dP as a function of the differential of volume dV and temperature dT. We only need to use the general formula (2.95) and the relation

$$P(V, T) = R\frac{T}{V}, \tag{2.99}$$

which follows from (2.98) and we obtain

$$dP = \left(\frac{\partial P}{\partial V}\right)_T dV + \left(\frac{\partial P}{\partial T}\right)_V dT = R\left(\frac{-T}{V^2}\right) dV + \left(\frac{R}{V}\right) dT. \tag{2.100}$$

The sub-indices V and T in the partial derivatives above only indicate which variable remains constant. For example $(\partial P/\partial V)_T$ is the partial derivative of the pressure with respect to the volume at constant temperature.

Example 2: Use differentials to estimate the value of $\sqrt{27}\sqrt[3]{1021}$.

Let us start by defining the function

$$f(x,y) = \sqrt{x}\sqrt[3]{y}. \tag{2.101}$$

We can easily obtain the value of this function at the point $(x,y) = (25,1000)$,

$$f(25,1000) = 5 \times 10 = 50. \tag{2.102}$$

We can now exploit the fact that the point at which we want to compute the value of $f(x,y)$ is very close to $(25,1000)$. In other words we can define

$$(27,1021) = (x+\Delta x, y+\Delta y) = (25+2, 1000+21), \tag{2.103}$$

hence identifying $\Delta x \simeq dx = 2$ and $\Delta y \simeq dy = 21$. Employing now our formula (2.95) we have that

$$df = f_x dx + f_y dy = \frac{1}{2}\frac{\sqrt[3]{y}}{\sqrt{x}}dx + \frac{1}{3}\frac{\sqrt{x}}{y^{2/3}}dy, \tag{2.104}$$

and evaluating this formula with $x = 25$, $y = 1000$, $dx = 2$ and $dy = 21$ we obtain

$$df = \frac{1}{2}\frac{10}{5}2 + \frac{1}{3}\frac{5}{100}21 = 2 + \frac{7}{20} = 2.35. \tag{2.105}$$

Therefore, the approximate value of $f(27,1021)$ is given by

$$f(27,1021) \simeq f(25,1000) + df = 52.35. \tag{2.106}$$

Now we can check if this approximation is good by calculating exactly the value

$$f(27,1021) = \sqrt{27}\sqrt[3]{1021} = 52.3227\ldots \simeq 52.32, \tag{2.107}$$

and so the approximation is actually very good!

It is now easy to generalize the concept of differential to functions of n variables:

Definition: The total differential df of a function of n variables $f(x_1,\ldots,x_n)$ with continuous 1st order partial derivatives $f_{x_1}\ldots f_{x_n}$ is given by

$$df = f_{x_1}dx_1 + f_{x_2}dx_2 + \ldots + f_{x_n}dx_n. \tag{2.108}$$

The object df is frequently called the **1st-order differential of the function** f. The reason is that, we can define **higher order differentials** in the following way:

Definition: Consider a function $f(x,y)$ of two real variables with continuous 1st- and 2nd-order partial derivatives. We define the 2nd-order differential of f and we write it as d^2f as

$$\begin{aligned} d^2 f &= d\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = d(f_x dx + f_y dy) \\ &= (df_x)dx + (df_y)dy = (f_{xx}dx + f_{yx}dy)dx + (f_{xy}dx + f_{yy}dy)dy \\ &= f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2, \end{aligned} \tag{2.109}$$

where we assumed $f_{xy} = f_{yx}$. An equivalent way to write this result is

$$d^2 f = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^2 f, \tag{2.110}$$

where we identify

$$\left(\frac{\partial}{\partial x} \right)^2 = \frac{\partial^2}{\partial x^2} \quad \text{and} \quad \left(\frac{\partial}{\partial y} \right)^2 = \frac{\partial^2}{\partial y^2}. \tag{2.111}$$

Now we can generalize the previous formula to the n-th differential of a function $f(x, y)$ as

Definition: The n-th differential of a function of two variables $f(x, y)$ with continuous n-th order partial derivatives is given by

$$d^n f = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n f. \tag{2.112}$$

This formula can be proven by induction (which means that assuming it works for $d^{n-1} f$ we can prove that it works for $d^n f$). For that proof it is important to notice that the operator above admits the binomial expansion

$$\left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\frac{\partial^k}{\partial x^k} \right) \left(\frac{\partial^{n-k}}{\partial y^{n-k}} \right) dy^{n-k} dx^k, \tag{2.113}$$

with

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}. \tag{2.114}$$

2.3.4 Functions of several variables defined implicitly

As usual let us start by looking at the easiest example, namely functions of one variable.

Definition: We say that the function $y = f(x)$ is defined implicitly if y and x are related by an equation of the type

$$\Phi(x, y) = 0, \tag{2.115}$$

and there is no possibility of obtaining $y = f(x)$ explicitly from the constraint (2.115).

Note: Notice however that the equation (2.115) allows us to obtain the value of y for a given value of x (at least numerically), even though it does not allow us to know $f(x)$ for arbitrary x.

It is easier to understand this definition with some examples:

Example 1: Let

$$\Phi(x, y = f(x)) = \log(x + y) - \sin(x + y) = 0. \tag{2.116}$$

The constraint $F(x, y) = 0$ gives us a relation between x and y, however we are not able to obtain y as a function of x from this equation. Therefore the function $y = f(x)$ is defined implicitly through the constraint $\Phi(x, y) = 0$.

Suppose now that we would like to know the value of y for $x = 0$. In this case we have

$$\Phi(0, y) = 0 = \log(y) - \sin(y) \quad \Rightarrow \quad \log(y) = \sin(y), \tag{2.117}$$

and this equation can be solved numerically by finding the points at which the curves $\log(y)$ and $\sin(y)$ meet. The solution is approximately $y = f(0) \simeq 2.22$.

Example 2: Let

$$G(x, y = f(x)) = e^{xy} - x^2 y \log(y) - 1 = 0. \tag{2.118}$$

Again $y = f(x)$ is an implicit function of x, since there is no way from the constraint $G(x, y) = 0$ to obtain y as function of x explicitly.

We can however obtain the value of y for fixed values of x, for example at $x = 1$ we have

$$e^y = y \log(y) + 1. \tag{2.119}$$

The approximate numerical solution of this equation is $y = f(1) \simeq 1.16$.

The definition for functions of one variable extends easily to functions of several variables. For example, for functions of two variables $z = f(x, y)$ we have:

Definition: A function of two variables $z = f(x, y)$ is said to be an implicit function of the independent variables x and y if z, x and y are related to each other by a constraint of the type

$$\Phi(x, y, z) = 0, \tag{2.120}$$

from which z can not be expressed explicitly in terms of x and y. As for the case of functions of one variable, it will still be possible to obtain z from the condition above for fixed values of (x, y).

Example: Consider the function $z = f(x, y)$ defined by

$$x^3 + 2x^2yz + \sin z - 1 = 0, \tag{2.121}$$

It is not possible to obtain $z = f(x, y)$ from the equation above for generic values of x, y. However we can solve it for given values of x and y. For example, take $x = y = 1$

$$2z + \sin(z) = 0, \tag{2.122}$$

which has solution $z = f(1, 1) = 0$.

Partial derivatives of implicit functions: The main problem we want to address in this section is the following: **Given a function of several variables defined implicitly, how can we obtain the partial derivatives of this function?** Let us consider an implicit function of two variables $z = f(x, y)$ and assume the existence of a constraint

$$\Phi(x, y, z) = 0, \tag{2.123}$$

which relates the function z to the two independent variables x and y. Since $\Phi = 0$ it is clear that also its total differential $d\Phi = 0$ must vanish. However the total differential is by definition

$$d\Phi = \left(\frac{\partial \Phi}{\partial x}\right) dx + \left(\frac{\partial \Phi}{\partial y}\right) dy + \left(\frac{\partial \Phi}{\partial z}\right) dz = 0, \tag{2.124}$$

and in addition, z is a function of x and y, therefore its differential is given by

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy. \tag{2.125}$$

If we substitute (2.125) into (2.124) we obtain the equation

$$d\Phi = 0 = \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial y}\right) dy. \tag{2.126}$$

Since x and y are independent variables, equation (2.126) implies that each of the factors has to vanish separately, that is

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial y} = 0. \tag{2.127}$$

Therefore we obtain,

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial \Phi}{\partial x}\right)}{\left(\frac{\partial \Phi}{\partial z}\right)} = -\frac{\Phi_x}{\Phi_z}, \tag{2.128}$$

$$\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial F}{\partial y}\right)}{\left(\frac{\partial \Phi}{\partial z}\right)} = -\frac{\Phi_y}{\Phi_z}. \tag{2.129}$$

Example 1: Consider again the function of the previous example

$$\Phi(x, y, z) = x^3 + 2x^2yz + \sin z - 1 = 0, \tag{2.130}$$

where z is an implicit function of the independent variables x and y. Obtain the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$.

Here we must only apply the equations (2.128) and (2.129). For that we need the partial derivatives

$$\Phi_x = 3x^2 + 4xyz, \qquad \Phi_y = 2x^2 z \quad \text{and} \quad \Phi_z = 2x^2y + \cos z, \tag{2.131}$$

and so we obtain

$$\frac{\partial z}{\partial x} = -\frac{\Phi_x}{\Phi_z} = -\frac{3x^2 + 4xyz}{2x^2y + \cos z}, \qquad \frac{\partial z}{\partial y} = -\frac{\Phi_y}{\Phi_z} = -\frac{2x^2 z}{2x^2y + \cos z}. \tag{2.132}$$

Notice that the derivatives are still functions of z and therefore if we want to obtain their value at a particular point (x, y) we need to solve first (2.130) and then substitute the corresponding value of z in (2.132). For example we found in the previous example (see previous page) that $z = f(1, 1) = 0$, therefore the partial derivatives at the point $(1, 1)$ are

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1)} = -\frac{\Phi_x(1,1)}{\Phi_z(1,1)} = 1, \qquad \left.\frac{\partial z}{\partial y}\right|_{(1,1)} = -\frac{\Phi_y(1,1)}{\Phi_z(1,1)} = 0. \tag{2.133}$$

Example 2: Consider now the same function of example 1 and assume in addition that the variables x and y are functions of two other variables u and v as

$$x = \frac{u^2 - v^2}{2} \qquad \text{and} \qquad y = uv. \tag{2.134}$$

Obtain the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$.

In this example we have to combine the results (2.128)-(2.129) with the chain rules we learnt in previous sections. The chain rule tells us that

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}, \end{aligned} \tag{2.135}$$

and the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ were computed in (2.132). Therefore, we only need to compute

$$\frac{\partial x}{\partial u} = u, \quad \frac{\partial x}{\partial v} = -v, \quad \frac{\partial y}{\partial u} = v \quad \text{and} \quad \frac{\partial y}{\partial v} = u. \tag{2.136}$$

Plugging (2.132) and (2.136) in (2.135) we obtain

$$\frac{\partial z}{\partial u} = -\frac{3x^2 + 4xyz}{2x^2y + \cos z}u - \frac{2x^2 z}{2x^2y + \cos z}v, \tag{2.137}$$

$$\frac{\partial z}{\partial v} = \frac{2x^2 + 4xyz}{2x^2y + \cos z}v - \frac{2x^2 z}{2x^2y + \cos z}u, \tag{2.138}$$

and if we replace x and y in terms of u and v we obtain,

$$\frac{\partial z}{\partial u} = \frac{(u^2-v^2)/2}{(u^2-v^2)^2uv+2\cos z}\left[-(3(u^2-v^2)+8uvz)u-(u^2-v^2)zv\right], \quad (2.139)$$

$$\frac{\partial z}{\partial v} = \frac{u^2-v^2}{(u^2-v^2)^2uv+2\cos z}\left[(u^2-v^2+4uvz)v-(u^2-v^2)zu\right]. \quad (2.140)$$

2.4 Local properties of functions of several variables

In this section we will learn how to address three kinds of problems which are of great importance in the field of applied mathematics: how to obtain the **approximate value** of functions of several variables near a point in their domain, how to obtain and classify the **extreme values** (maxima, minima and saddle points) of functions of several variables and finally, how to solve so-called **constrained extrema problems**. We will start by addressing the first of these questions:

2.4.1 Taylor series expansions

A good way of obtaining the value of a function near a point at which the value of the function and its derivatives are known is by means of Taylor's series expansion. Let us recall it for functions of one variable:

Definition: Let $f(x)$ be a function of one variable with continuous derivatives of all orders at a the point x_0, then the series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \tag{2.141}$$

is called the **Taylor series expansion of f about $x = x_0$**. If $x_0 = 0$ the term **Maclaurin series** is usually employed in place of Taylor series.

<u>**Note:**</u> In practice when employing Taylor's series we will only consider the first contributions to the sum above and suppose that they provide already a good enough approximation. In fact, if we 'cut' the series (2.141) at $k = n$ we will obtain the 'best' order-n polynomial approximation of the function $f(x)$ near the point x_0.

The Taylor series expansion can be easily generalized to functions of more than one variable. Let us as usual consider the case of functions of two variables:

<u>**Definition:**</u> Let $f(x, y)$ be a function of two real variables which is continuous at a certain point (x_0, y_0) and such that all its partial derivatives are also continuous at that point. Then the Taylor series expansion of $f(x, y)$ about the point (x_0, y_0) can be obtained exactly in the same way as for functions of one variable. We can first apply (2.141) to expand the function on the variable x about x_0 keeping y fixed:

$$f(x, y) = \sum_{k=0}^{\infty} \frac{1}{k!} \left.\frac{\partial^k f}{\partial x^k}\right|_{(x_0, y)} (x - x_0)^k. \tag{2.142}$$

We can now take the expansion (2.142) and treat it as a function of y. If we do so, we can use again Taylor's expansion for functions of one variable on (2.142) and expand it about the point $y = y_0$,

$$f(x, y) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{p!\,k!} \left.\frac{\partial^p}{\partial y^p}\left(\frac{\partial^k f}{\partial x^k}\right)\right|_{(x_0, y_0)} (x - x_0)^k (y - y_0)^p. \tag{2.143}$$

Note: Notice that if we would have first expanded about y_0 and then about x_0 the derivatives with respect to x and y in (2.143) would appear in the reverse order. However, since we assume that f is continuous and has continuous partial derivatives of all orders at the point (x_0, y_0) this implies that

$$\frac{\partial^p}{\partial y^p}\left(\frac{\partial^k f}{\partial x^k}\right)\bigg|_{(x_0,y_0)} = \frac{\partial^k}{\partial x^k}\left(\frac{\partial^p f}{\partial y^p}\right)\bigg|_{(x_0,y_0)}, \tag{2.144}$$

and therefore both formulae are equivalent.

An alternative version of Taylor's formula: Let us consider the first terms on the expansion (2.143). They are

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ f_{yx}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 \\ &+ \frac{1}{2}f_{xx}(y_0,x_0)(x-x_0)^2 + \cdots \\ &= \varphi^{(0)}(x_0,y_0) + \varphi^{(1)}(x_0,y_0) + \frac{1}{2}\varphi^{(2)}(x_0,y_0) + \cdots \end{aligned} \tag{2.145}$$

with

$$\varphi^{(n)}(x_0,y_0) = \left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^n f(x,y)\bigg|_{(x_0,y_0)}. \tag{2.146}$$

It is easy to prove that the appearance of the operator $\varphi^{(n)}(x_0, y_0)$ extends to all other terms in the Taylor expansion (in fact it can be proven by induction, in the same way described after (2.112)). This means that we can write (2.143) as

$$f(x,y) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_0,y_0)}{n!}. \tag{2.147}$$

Note: Notice that the operation (2.146) means that after expanding the n-power we act first with the partial derivatives on f and then take those derivatives at the point (x_0, y_0). For example

$$\begin{aligned} \varphi^{(2)}(x_0,y_0) &= \left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^2 f(x,y)\bigg|_{(x_0,y_0)} \qquad (2.148) \\ &= \left((x-x_0)^2\frac{\partial^2}{\partial x^2} + (y-y_0)^2\frac{\partial^2}{\partial y^2} + 2(y-y_0)(x-x_0)\frac{\partial^2}{\partial y\partial x}\right) f(x,y)\bigg|_{(x_0,y_0)} \\ &= (x-x_0)^2 f_{xx}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0) + 2(y-y_0)(x-x_0)f_{xy}(x_0,y_0). \end{aligned}$$

Let us see the working of these formulae with one example:

Example: Let

$$f(x,y) = x^2y^3. \tag{2.149}$$

Obtain the Taylor expansion of this function about the point $(1, 1)$ including up to second order terms.

By second-order terms it is meant that we take the terms on the Taylor expansion until second-order partial derivatives. That means that we need to calculate the following sum

$$\begin{aligned} f(x,y) &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) + \frac{1}{2} f_{xx}(1,1)(x-1)^2 \\ &+ \frac{1}{2} f_{yy}(1,1)(y-1)^2 + f_{xy}(1,1)(x-1)(y-1) + \cdots . \end{aligned} \quad (2.150)$$

Therefore the first thing we need to compute are the 1st- and 2nd-order partial derivatives of f,

$$f_x = 2xy^3 \qquad f_y = 3x^2y^2 \qquad f_{xx} = 2y^3, \quad (2.151)$$

$$f_{yy} = 6x^2y \qquad f_{xy} = f_{yx} = 6xy^2. \quad (2.152)$$

Therefore we have

$$f(1,1) = 1 \qquad f_x(1,1) = 2 \qquad f_y(1,1) = 3, \quad (2.153)$$

$$f_{xx}(1,1) = 2 \qquad f_{yy}(1,1) = 6 \qquad f_{xy}(1,1) = 6. \quad (2.154)$$

Thefore the expansion (2.150) is given by

$$\begin{aligned} f(x,y) &\simeq 1 + 2(x-1) + 3(y-1) + (x-1)^2 + 3(y-1)^2 + 6(x-1)(y-1) \\ &= 6 - 6x - 9y + x^2 + 3y^2 + 6xy. \end{aligned} \quad (2.155)$$

We can actually check how good this approximation is near the point $(1,1)$ by plotting the exact function and the approximate function (2.155):

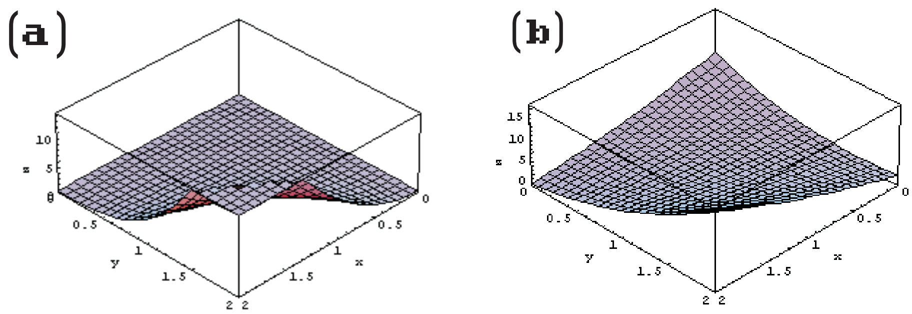


Figure 8: *The function $f(x,y) = x^2y^3$ (a) and its Taylor approximation (b).*

In the picture above you can see that near $(1,1)$ both functions are very similar. Therefore the approximation (2.155) is quite good there. However if we go a bit far from $(1,1)$, for example the point $(0,0)$ both functions are already very different. In fact function (a) takes the value 0 at (0,0), whereas function (b) is 6 at the same point!.

As we see from the example, it is common to take only a few terms of the Taylor expansion of a function around a certain point. It is therefore convenient to have a formula which tells us precisely the order of magnitude of the error we make when we take only n terms in the expansion (2.147). This formula is the so-called **Taylor expansion formula with Lagrange's remainder** and has the following form:

Definition: Given a function $f(x,y)$ with Taylor expansion (2.147) we can write

$$f(x,y) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(x_0,y_0)}{k!} + R_n(\tilde{x},\tilde{y}), \tag{2.156}$$

where $R_n(\tilde{x},\tilde{y})$ is Lagrange's remainder or error term and is given by

$$R_n(\tilde{x},\tilde{y}) = \frac{\varphi^{(n+1)}(\tilde{x},\tilde{y})}{(n+1)!}, \tag{2.157}$$

where $(\tilde{x},\tilde{y})$ is a point such that $\tilde{x}$ is a number between x and x_0 and $\tilde{y}$ is a number between y and y_0.

Example: Estimate the value of the remainder of the Taylor expansion given in the previous example at the point $(1.1, 0.9)$.

In the previous example we considered the function $f(x,y) = x^2y^3$ and carried out its Taylor expansion around the point $(1,1)$ up to 2nd-order terms. We obtained the result

$$f(x,y) \simeq 6 - 6x - 9y + x^2 + 3y^2 + 6xy, \tag{2.158}$$

which means that

$$f(1.1, 0.9) \simeq 0.88. \tag{2.159}$$

The problem is asking us what error we make when we approximate $f(1.1, 0.9)$ by the value (2.159). The answer to this question is given by computing the remainder of the Taylor expansion at the point $(1.1, 0.9)$. According to our definition of the remainder, we need to compute

$$R_2(\tilde{x},\tilde{y}) = \frac{\varphi^{(3)}(\tilde{x},\tilde{y})}{3!}, \tag{2.160}$$

where $(\tilde{x},\tilde{y})$ is by definition a point in between $(1,1)$ and $(1.1, 0.9)$. We have also seen that $\varphi^{(3)}$ is given by

$$\begin{aligned}
\varphi^{(3)}(\tilde{x},\tilde{y}) &= \left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^3 f(x,y)\Big|_{(\tilde{x},\tilde{y})} \\
&= \Bigg((x-x_0)^3\frac{\partial^3}{\partial x^3} + (y-y_0)^3\frac{\partial^3}{\partial y^3} + 3(x-x_0)^2(y-y_0)\frac{\partial^3}{\partial x^2\partial y} \\
&\quad +3(x-x_0)(y-y_0)^2\frac{\partial^3}{\partial x\partial y^2}\Bigg) f(x,y)\Big|_{(\tilde{x},\tilde{y})} \\
&= (x-x_0)^3 f_{xxx}(\tilde{x},\tilde{y}) + (y-y_0)^3 f_{yyy}(\tilde{x},\tilde{y}) + 3(x-x_0)^2(y-y_0)f_{xxy}(\tilde{x},\tilde{y}) \\
&\quad +3(x-x_0)(y-y_0)^2 f_{xyy}(\tilde{x},\tilde{y}).
\end{aligned} \tag{2.161}$$

where we have used the fact that the order of the derivatives does not matter if f and its derivatives are continuous. This allows us to assume $f_{xxy} = f_{xyx} = f_{yxx}$ and $f_{yyx} = f_{yxy} = f_{xyy}$. Notice that the coordinates of all three points (x,y), (x_0,y_0) and $(\tilde{x},\tilde{y})$ are involved in the formula!

In order to evaluate the remainder we need to obtain all 3th order partial derivatives of the function x^2y^3. They are given by

$$f_{xxx}(\tilde{x},\tilde{y}) = 0, \quad f_{yxx}(\tilde{x},\tilde{y}) = 6\tilde{y}^2, \quad f_{yyy}(\tilde{x},\tilde{y}) = 6\tilde{x}^2, \quad f_{xyy}(\tilde{x},\tilde{y}) = 12\tilde{x}\tilde{y}, \tag{2.162}$$

Thus we obtain

$$\varphi^{(3)}(\tilde{x},\tilde{y}) \quad = \quad 6(y-1)^3\tilde{x}^2 + 18\tilde{y}^2(x-1)^2(y-1) + 36\tilde{x}\tilde{y}(x-1)(y-1)^2. \quad (2.163)$$

For $(x, y) = (1.1, 0.9)$ the remainder becomes,

$$R_2(\tilde{x},\tilde{y}) = \frac{\varphi^{(3)}(\tilde{x},\tilde{y})}{6} = (-0.1)^3\tilde{x}^2 + 3\tilde{y}^2(0.1)^2(-0.1) + 6\tilde{x}\tilde{y}(0.1)(-0.1)^2. \quad (2.164)$$

As we said before $(\tilde{x},\tilde{y})$ is some point lying between $(1, 1)$ and $(1.1, 0.9)$, which means that

$$1 < \tilde{x} < 1.1 \qquad \text{and} \qquad 0.9 < \tilde{y} < 1. \quad (2.165)$$

Let us for example take the middle point $\tilde{x} = 1.05$ and $\tilde{y} = 0.95$ and substitute into the remainder,

$$R_2(1.05, 0.95) = (-0.1)^3(1.05)^2+3(0.95)^2(0.1)^2(-0.1)+6(1.05)(0.95)(0.1)(-0.1)^2 = 0.002175. \quad (2.166)$$

we find that the maximum error we make should be smaller than 0.002175. We can now check if this is true by computing the exact value of

$$f(1.1, 0.9) = (1.1)^2(0.9)^3 = 0.88209, \quad (2.167)$$

and comparing it to (2.159). We find that the difference between the two values is 0.00209 which is indeed smaller than 0.002175.

2.4.2 Classification of extremes of functions of two variables

In this section we are going to study the conditions under which a point (x_0, y_0) is a **maximum, a minimum or a saddle point** of a function of two variables $f(x, y)$. Similarly as for functions of one variable, the values of 1st-order and 2nd-order partial derivatives will determine if a certain point is an extreme of the function f.

Definition: If a function of two variables $f(x, y)$ has a maximum or a minimum at a point (x_0, y_0), then its 1st order partial derivatives vanish at that point

$$(x_0, y_0) \text{ maximum or minimum of } f(x, y) \Rightarrow f_x(x_0, y_0) = f_y(x_0, y_0) = 0. \tag{2.168}$$

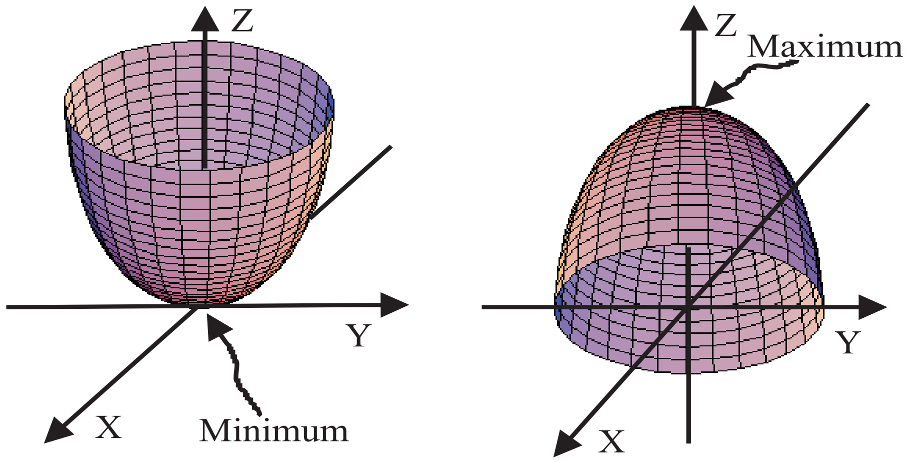


Figure 9: *A minimum and a maximum*

Note: Note that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ **are necessary conditions** for (x_0, y_0) to be a maximum or a minimum of f but they are **not sufficient conditions**.

Classification of stationary points: Let us consider a function $f(x, y)$ such that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. In this case if we Taylor expand the function around the point (x_0, y_0) the 1st-order terms vanish and therefore the 2nd-order terms give the first contribution to the Taylor expansion. This implies

$$f(x, y) = f(x_0, y_0) + \frac{1}{2}\left[A(x - x_0)^2 + 2B(x - x_0)(y - y_0) + C(y - y_0)^2\right], \tag{2.169}$$

where we called

$$A = f_{xx}(x_0, y_0), \qquad B = f_{xy}(x_0, y_0), \qquad C = f_{yy}(x_0, y_0). \tag{2.170}$$

Here we have assumed that f is a continuous function at (x_0, y_0) and therefore $f_{xy} = f_{yx}$ and that its 2nd-order partial derivatives exist and are continuous in a neighbourhood of (x_0, y_0). If we introduce

$$\Delta f = f(x, y) - f(x_0, y_0), \tag{2.171}$$

and we parameterize

$$x - x_0 = r \cos \alpha, \qquad y - y_0 = r \sin \alpha, \tag{2.172}$$

in terms of two new variables α, r we obtain

$$\Delta f = \frac{1}{2}\left[Ar^2\cos^2\alpha + 2Br^2\sin\alpha\cos\alpha + Cr^2\sin^2\alpha\right]. \tag{2.173}$$

Now the main idea is that depending on the sign of Δf we will be able to say if the function has a maximum, a minimum or a saddle point at (x_0, y_0). We can formulate this more precisely as follows:

Maximum: If $\Delta f < 0$ for all values of α, then (x_0, y_0) is a maximum of the function f.

Minimum: If $\Delta f > 0$ for all values of α, then (x_0, y_0) is a minimum of the function f.

Saddle point: If the sign of Δf changes at a particular value of $\alpha = \alpha_0$, then (x_0, y_0) is a saddle point of the function f.

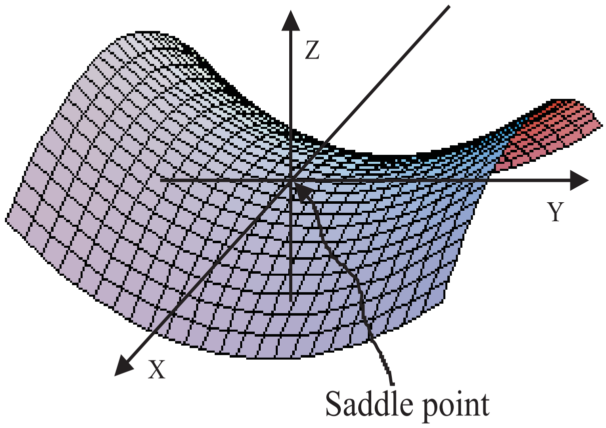


Figure 10: *Saddle point*

Let us now consider all possible situations that can arise depending on the values of A, B and C:

1. If $A \neq 0$, then we can rewrite Δf in the following equivalent form

$$\Delta f = \frac{r^2(A\cos\alpha + B\sin\alpha)^2 + r^2(AC - B^2)\sin^2\alpha}{2A}. \tag{2.174}$$

We can now consider two cases:

- if $AC - B^2 > 0$, then the sign of Δf is determined by the sign of A. Therefore provided that $AC - B^2 > 0$ we have

$$\textbf{maximum if } A < 0 \quad \textbf{and} \quad \textbf{minimum if } A > 0. \tag{2.175}$$

- if $AC - B^2 < 0$, then the numerator will change sign at some particular value of $\alpha = \alpha_0$, therefore we will have a **saddle point**.
- if $AC - B^2 = 0$, then the sign of Δf does not change but there is a value of α for which $\Delta f = 0$. This value corresponds to $\cot\alpha = -B/A$. In this case the situation is **inconclusive** (we are not able to say if the point is a maximum, a minimum or a saddle point).

2. If $A = 0$ and $B \neq 0$, then

$$\Delta f = \frac{r^2}{2}\left[2B \sin\alpha \cos\alpha + C \sin^2\alpha\right]. \tag{2.176}$$

   Since $\sin^2\alpha$ is always positive but the term $2B\sin\alpha\cos\alpha$ can be positive or negative, there is a value of α at which the sign of Δf changes. Therefore we have a **saddle point**.

3. If $A = B = 0$, then

$$\Delta f = Cr^2 \sin^2\alpha. \tag{2.177}$$

   Therefore Δf does not change sign but becomes zero as $\alpha \to 0$. Therefore the situation is **inconclusive**.

All these results can be summarized in the following table:

$AC - B^2$	> 0	< 0	0
$A > 0$	minimum	saddle point	inconclusive
$A < 0$	maximum	saddle point	inconclusive
$A = 0$		saddle point	inconclusive

Example 1: Classify the stationary points of the function:

$$f(x, y) = 1 - x^2 - y^2. \tag{2.178}$$

We start by computing the 1st-partial derivatives:

$$f_x = -2x, \qquad f_y = -2y. \tag{2.179}$$

We immediately see that $f_x(0,0) = f_y(0,0) = 0$. Therefore we need to obtain the 2nd-order partial derivatives. They are

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = f_{yx} = 0. \tag{2.180}$$

Since they take constant values, these are also the values at the point $(0,0)$. Therefore

$$AC - B^2 = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 4 > 0. \tag{2.181}$$

Now, as we have seen the nature of the point is determined by the sign of $A = -2 < 0$. Therefore $(0,0)$ is a maximum of f (see the picture below).

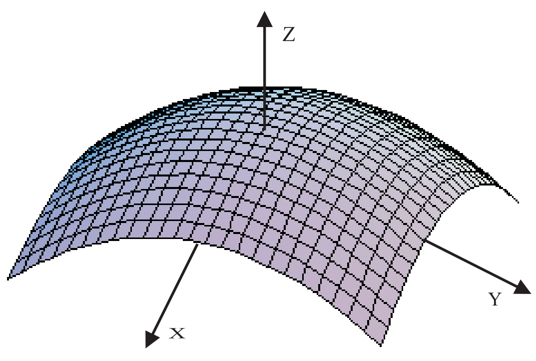


Figure 11: *The function* $1 - x^2 - y^2$

Example 2: Let

$$f(x,y) = xye^{-(x^2+y^2)/2}. \tag{2.182}$$

Classify its stationary points.

As usual we commence by finding the points at which the 1st-order partial derivatives are zero. In this case we have

$$f_x = y(1-x^2)e^{-(x^2+y^2)/2}, \qquad f_y = x(1-y^2)e^{-(x^2+y^2)/2}. \tag{2.183}$$

We find that $f_x = 0$ for $x = \pm 1$ and $y = 0$ and $f_y = 0$ for $y = \pm 1$ and $x = 0$. Therefore the two partial derivatives vanish at the following 5 points:

$$(1,1) \quad (1,-1) \quad (-1,1) \quad (-1,-1) \quad (0,0). \tag{2.184}$$

We compute now the 2nd-order partial derivatives:

$$\begin{aligned} f_{xx} &= xy(x^2-3)e^{-(x^2+y^2)/2}, \quad f_{yy} = xy(y^2-3)e^{-(x^2+y^2)/2}, \\ f_{xy} &= f_{yx} = (1-x^2)(1-y^2)e^{-(x^2+y^2)/2}, \end{aligned} \tag{2.185}$$

and study each of the points separately:

The point $(1,1)$**:** At this point

$$A = -2/e, \qquad B = 0, \qquad C = -2/e. \tag{2.186}$$

Therefore $A < 0$ and $AC - B^2 = 4/e^2 > 0$. This point is a maximum.

The point $(-1,-1)$**:** At this point

$$A = -2/e, \qquad B = 0, \qquad C = -2/e. \tag{2.187}$$

Therefore $A < 0$ and $AC - B^2 = 4/e^2 > 0$. This point is also a maximum.

The point $(1,-1)$**:** At this point

$$A = 2/e, \qquad B = 0, \qquad C = 2/e. \tag{2.188}$$

Therefore $A > 0$ and $AC - B^2 = 4/e^2 > 0$. This point is a minimum.

The point $(-1,1)$**:** At this point

$$A = 2/e, \qquad B = 0, \qquad C = 2/e. \tag{2.189}$$

Therefore $A > 0$ and $AC - B^2 = 4/e^2 > 0$. This point is also a minimum.

The point $(0,0)$**:** At this point

$$A = 0, \qquad B = 1, \qquad C = 0. \tag{2.190}$$

Therefore $A = 0$ and $B \neq 0$. This point is a saddle point.

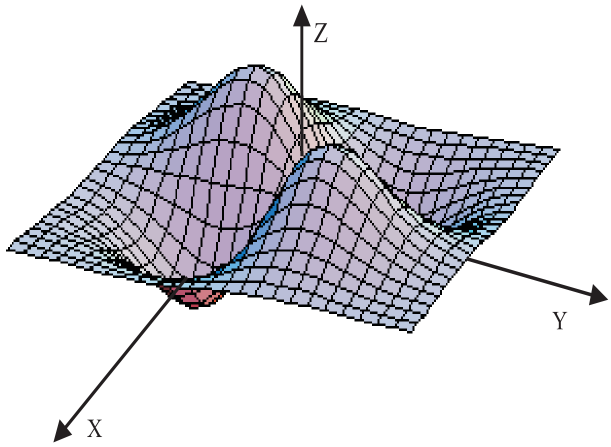


Figure 12: *The function* $xye^{-(x^2+y^2)/2}$

2.4.3 Constraints via Lagrange multipliers

In this section we will see a particular method to solve so-called problems of constrained extrema. There are two kinds of typical problems:

Finding the shortest distance from a point to a plane: Given a plane

$$Ax + By + Cz + D = 0, \tag{2.191}$$

obtain the shortest distance from a point (x_0, y_0, z_0) to this plane. In this case we have to minimize the function

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, \tag{2.192}$$

with the constraint that the point (x, y, z) is in the plane (2.191).

Shortest distance from a point to a generic surface: This is a more general problem where the equation of a three dimensional surface is given,

$$\phi(x, y, z) = 0, \tag{2.193}$$

and we are asked to obtain the shortest distance from a point (x_0, y_0, z_0) to this surface. Again we need to minimize the function distance including the constraint that the points we want to consider all belong to the surface (2.193).

The method of Lagrange multipliers provides an easy way to solve this kind of problems. Assume that $f(x, y, z)$ is the function we want to minimize (in the examples it would be the distance). Then if the function has a minimum at a point (x_0, y_0, z_0) its first order differential vanishes at that point:

$$df = f_x dx + f_y dy + f_z dz = 0. \tag{2.194}$$

Suppose that

$$\phi(x, y, z) = 0, \tag{2.195}$$

is the constraint (in the previous examples the equation of a certain surface). Then it follows trivially that

$$d\phi = \phi_x dx + \phi_y dy + \phi_z dz = 0, \tag{2.196}$$

and therefore we can also write that

$$d(f + \lambda\phi) = df + \lambda d\phi = 0, \tag{2.197}$$

where λ is an arbitrary constant which we call **Lagrange's multiplier**. The previous equation is equivalent to

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right) dz = 0. \tag{2.198}$$

Due to the constraint (2.195), z is an implicit function of the independent variables x, y. Since λ is arbitrary we can choose it to have a particular value. For example, let us choose it so that

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0. \tag{2.199}$$

In that case equation (2.198) reduces to

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right) dy = 0, \tag{2.200}$$

and since x and y are independent variables, there should not be a relationship between their differentials, so the only way to solve the equation above is to take

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0. \tag{2.201}$$

Therefore, we have now a set of 4 equations, namely

$$\phi(x, y, z) = 0, \tag{2.202}$$
$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0, \tag{2.203}$$
$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0, \tag{2.204}$$
$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0, \tag{2.205}$$

which determine completely the 4 unknowns of the problem, namely the coordinates of the point (x, y, z) which is closest to the point (x_0, y_0, z_0) and the value of λ.

Remark: Notice that we can use the same method for problems involving only two variables. In that case we will have three equations, instead of four.

Example 1: Using the method of Lagrange multipliers find the shortest distance from the point $(0, 0, 1)$ to the surface $yx + yz + xz = 0$.

Following the general scheme given above we can identify the constraint

$$\phi(x, y, z) = yx + yz + xz = 0, \tag{2.206}$$

and the function we want to minimize is

$$f(x, y, z) = x^2 + y^2 + (z - 1)^2. \tag{2.207}$$

Therefore we can directly apply the method we have just seen and reduce the problem to the solution of the 4 equations (2.202)-(2.205). For that we need to evaluate the partial derivatives

$$f_x = 2x, \qquad f_y = 2y, \qquad f_z = 2(z - 1), \tag{2.208}$$
$$\phi_x = z + y, \qquad \phi_y = x + z, \qquad \phi_z = x + y. \tag{2.209}$$

And so we have to solve the equations

$$yx + yz + xz = 0, \tag{2.210}$$
$$2(z - 1) + \lambda(x + y) = 0, \tag{2.211}$$
$$2x + \lambda(z + y) = 0, \tag{2.212}$$
$$2y + \lambda(x + z) = 0. \tag{2.213}$$

From the last 3 equations we obtain:

$$\lambda = -\frac{2(z - 1)}{x + y} = -\frac{2x}{z + y} = -\frac{2y}{x + z}, \tag{2.214}$$

and we can rewrite the last equality as,

$$\frac{x}{z+y} = \frac{y}{x+z} \quad \Leftrightarrow \quad x(x+z) = y(y+z), \tag{2.215}$$

which is equivalent to

$$z(y-x) = x^2 - y^2 = (x-y)(x+y). \tag{2.216}$$

The last equation admits two solutions: $x = y$ and $z = -(x+y)$ with $x \neq y$.

Let us take the solution $x = y$ for now. If we put this into (2.212) we get that

$$z = -x\frac{2+\lambda}{\lambda}.$$

If we put this and into equation (2.210) we get:

$$x^2\left(1 - 2\frac{2+\lambda}{\lambda}\right) = 0, \tag{2.217}$$

which is solved by

$$\lambda = -4, \quad \text{or} \quad x = 0. \tag{2.218}$$

If $\lambda = -4$ then $z = -x/2$. Substituting in (2.211) with $x = y$ we obtain

$$x = -2/9 = y, \tag{2.219}$$

and $x = 1/9$. If we take the other solution to (2.217), namely $x = 0 = y$, then, substituting in (2.211) we obtain $z = 1$ and substituting this in (2.212) we obtain $\lambda = 0$.

Therefore we have the points $(x,y,z) = (0,0,1)$ and $(x,y,z) = (-2/9,-2/9,1/9)$. The fist solution is the same point whose distance to we wanted to minimize. Therefore we have discovered that the point $(0,0,1)$ is itself on the surface (2.135) and therefore the solution to our problem is the same point. The second solution we obtain must be therefore the point on the surface whose distance to $(0,0,1)$ is maximal. In fact the distance is

$$r = \sqrt{(2/9)^2 + (2/9)^2 + (8/9)^2} = \sqrt{72/81} = \frac{2\sqrt{2}}{3}. \tag{2.220}$$

Finally, let us go back to the line after (2.216). There we had obtained a second possible solution besides $x = y$. It was $z = -x - y$ with $x \neq y$. We must see if this gives any further solutions. If we substitute this in (2.211) we obtain $(x+y)(-2+\lambda) = 2$, that is $x + y = 2/(-2+\lambda)$. If we now take equations (2.212) and (2.213) and add them up, we obtain

$$2(x+y) + \lambda(y + 2z + x) = 0 \quad \Leftrightarrow \quad (x+y)(2-\lambda) = 0, \tag{2.221}$$

this gives an equation for λ,

$$\frac{(2-\lambda)}{-2+\lambda} = 0, \tag{2.222}$$

which does not make sense. Therefore the solution $z = -x - y$ is not valid.

Example 2: Using the method of Lagrange multipliers, determine the maximum of the function

$$f(x,y,z) = xyz, \tag{2.223}$$

subject to the condition

$$x^3 + y^3 + z^3 = 1, \tag{2.224}$$

with $x \geq 0, y \geq 0, z \geq 0$.

In this case our constraint is

$$\phi(x, y, z) = x^3 + y^3 + z^3 - 1 = 0, \tag{2.225}$$

and the corresponding partial derivatives of f and ϕ are

$$\begin{aligned} f_x &= zy, \qquad f_y = xz, \qquad f_z = xy, && (2.226) \\ \phi_x &= 3x^2, \qquad \phi_y = 3y^2, \qquad \phi_z = 3z^2. && (2.227) \end{aligned}$$

Therefore we need to solve the following system of equations

$$\begin{aligned} x^3 + y^3 + z^3 - 1 &= 0, && (2.228) \\ zy + \lambda 3x^2 &= 0, && (2.229) \\ zx + \lambda 3y^2 &= 0, && (2.230) \\ xy + \lambda 3z^2 &= 0. && (2.231) \end{aligned}$$

It is useful to try to see if there are any obvious solutions that we could get without too much work. For example, if we set any two variables to zero, then the remaining one will automatically need to be 1, from equation (2.228). This immediately gives us three solutions: the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. These solutions correspond to $\lambda = 0$. Another solution which is easy to see is to take $x = y = z$. Substituting this into (2.228) will give us

$$x = y = z = \sqrt[3]{\frac{1}{3}}, \tag{2.232}$$

and any of the other equations will tell us that $\lambda = -1/3$. These are in fact all the solutions to our problem. If we now compute

$$f(0, 0, 1) = f(1, 0, 0) = f(0, 1, 0) = 0, \tag{2.233}$$

and

$$f(\sqrt[3]{\frac{1}{3}}, \sqrt[3]{\frac{1}{3}}, \sqrt[3]{\frac{1}{3}}) = 1/3, \tag{2.234}$$

we find that f has minimum value at the points $(0, 0, 1), (1, 0, 0), (0, 1, 0)$ and is maximal at $(\sqrt[3]{\frac{1}{3}}, \sqrt[3]{\frac{1}{3}}, \sqrt[3]{\frac{1}{3}})$.

2.5 Integration of functions of several variables

In this section we will generalize the integration methods you know for functions of one variable to the case of functions of two and three variables. We will address mainly three types of problems: the problem of computing the area enclosed by a certain curve, the problem of computing the volume enclosed by a certain surface and the problem of performing changes of coordinates for integrals of functions of several variables.

2.5.1 Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

Volume under a surface: Let us consider the following typical problem: Calculate the volume under a surface $z = f(x, y)$ where (x, y) are points in a certain region R of the xy-plane. How can we solve this problem? The simplest method we can think of is to divide the region R into a net of small rectangles and construct above each of these rectangles a a column of hight $z = f(x, y)$

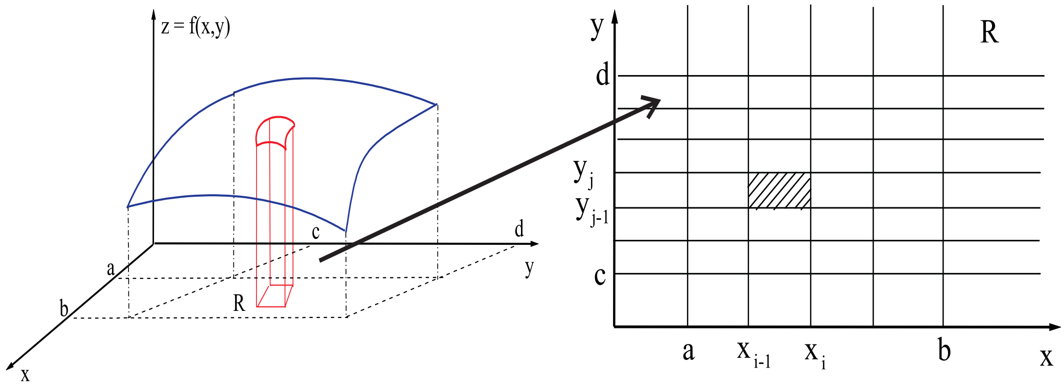


Figure 13: *Computation of the volume enclosed by the function $f(x, y)$. The figure on the r.h.s. shows the region R which we have divided into small rectangles. The dashed rectangle is the rectangle above which the column on the l.h.s. picture is constructed.*

Clearly the volume we want to compute would be the sum of the volumes of all columns like the one shown in figure 13. Suppose that the base of the column in figure 13 is the dashed rectangle of surface $(y_j - y_{j-1})(x_i - x_{i-1})$. Then, if we assume that the rectangle is actually very small (or the column very thin) we can approximate the volume of this column by

$$V_{ij} \simeq f(x_i, y_j)(y_j - y_{j-1})(x_i - x_{i-1}) = f(x_i, y_j)\Delta x_i \Delta y_j, \tag{2.235}$$

that is, we can suppose that the height of the column is $f(x_i, y_i)$ everywhere. Here we defined

$$\Delta x_i = x_i - x_{i-1}, \qquad \Delta y_j = y_j - y_{j-1}. \tag{2.236}$$

If we divide the region R, which in this case is

$$R = \{(x, y) : \quad a \leq x \leq b, \quad c \leq y \leq d\}, \tag{2.237}$$

into $n \times m$ rectangles (n intervals in the x-direction and m in the y-direction), we have that the total volume we want to compute is approximately given by

$$V \simeq \sum_{i=1}^{n} \sum_{j=1}^{m} V_{ij}. \tag{2.238}$$

Our computation of the volume will be more precise the smaller Δx_i and Δy_j are and become exact in the limit $\Delta x_i \to 0$ and $\Delta y_j \to 0$. In this limit the volume of the column in figure 13 becomes infinitesimal and is given by a differential dV and the increments (2.236) become also differentials. Thus

$$dV = f(x, y)dxdy, \tag{2.239}$$

and the total volume is obtained by replacing the sums in (2.238) by integrals

$$V = \int_{x=a}^{x=b} dx \int_{y=c}^{y=d} f(x, y)dy, \tag{2.240}$$

where the first integral runs between a and b and therefore corresponds to the variable x and the second integral runs between c and d and corresponds to the variable y.

Example 1: Compute the volume under the surface defined by

$$f(x, y) = 3x^2 + y^2, \quad \text{for} \quad 2 \le x \le 3 \quad \text{and} \quad 1 \le y \le 2. \tag{2.241}$$

Exploiting the formula we have just obtained we can compute the volume in two steps. First we integrate the function in the x-variable, treating y as a constant

$$\int_{x=2}^{x=3} (3x^2 + y^2)dx = \left[x^3 + xy^2\right]_2^3 = (27 + 3y^2) - (8 + 2y^2) = 19 + y^2. \tag{2.242}$$

The second and last step is to integrate in the y-variable

$$V = \int_{y=1}^{y=2} (19 + y^2)dy = \left[19y + \frac{y^3}{3}\right]_1^2 = \left(38 + \frac{8}{3}\right) - \left(19 + \frac{1}{3}\right) = \frac{64}{3}. \tag{2.243}$$

Example 2: Obtain the volume under the surface

$$f(x, y) = x + y, \quad \text{for} \quad 0 \le x \le 2 \quad \text{and} \quad 1 \le y \le 4. \tag{2.244}$$

We proceed like in the previous example. We carry out first the integral in x, treating y as a constant

$$\int_{x=0}^{x=2} (x + y)dx = \left[\frac{x^2}{2} + xy\right]_0^2 = (2 + 2y) - (0) = 2 + 2y. \tag{2.245}$$

Finally we carry out the integral in the variable y,

$$V = \int_{y=1}^{y=4} (2 + 2y)dy = \left[2y + y^2\right]_1^4 = (8 + 16) - (2 + 1) = 21. \tag{2.246}$$

It is important to notice that the order in which we carry out the integrals is irrelevant. For example, let us repeat this exercise by integrating firs in y, regarding x as a constant

$$\int_{y=1}^{y=4} (x + y)dy = \left[xy + \frac{y^2}{2}\right]_1^4 = (4x + 8) - (x + 1/2) = 3x + 15/2. \tag{2.247}$$

Now we can obtain the total volume by integrating in x

$$V = \int_{x=0}^{x=2} (3x + 15/2)dx = \left[3x^2/2 + 15x/2\right]_0^2 = (6 + 15) - (0) = 21. \tag{2.248}$$

Therefore, no matter the order in which we perform the integrations, we always obtain the same result.

Example 3: Obtain the volume under the surface

$$f(x, y) = x^2 y + y^2 x, \quad \text{for} \quad 0 \le x \le a \quad \text{and} \quad 0 \le y \le b. \tag{2.249}$$

As before, we start by integrating in one of the variables, for example the variable x, keeping y fixed

$$\begin{aligned} \int_{x=0}^{x=a} (x^2 y + y^2 x) dx &= \left[\frac{yx^3}{3} + \frac{x^2 y^2}{2}\right]_0^a = \left(\frac{ya^3}{3} + \frac{a^2 y^2}{2}\right) - (0) \\ &= \frac{ya^3}{3} + \frac{a^2 y^2}{2}. \end{aligned} \tag{2.250}$$

Finally we can take the previous result and integrate it with respect to y

$$\begin{aligned} V &= \int_{y=0}^{y=b} \left(\frac{ya^3}{3} + \frac{a^2 y^2}{2}\right) dy = \left[\frac{y^2 a^3}{6} + \frac{y^3 a^2}{6}\right]_0^b \\ &= \left(\frac{b^2 a^3}{6} + \frac{b^3 a^2}{6}\right) - (0) = \frac{b^2 a^3}{6} + \frac{b^3 a^2}{6}. \end{aligned} \tag{2.251}$$

Volume under a surface in a non-rectangular region of the xy-plane:
Let us now consider another typical sort of problem which involves the integration of a function of two variables. In the previous examples and exercises we have always considered integrals in a rectangular region of the xy-plane, which means that the values of x and y were in intervals of the type $x \in [a, b]$ and $y \in [c, d]$ (see figure 13). However in some cases we want to compute the volume under a function in a region of the xy-plane which is not a rectangle but some other kind of shape:

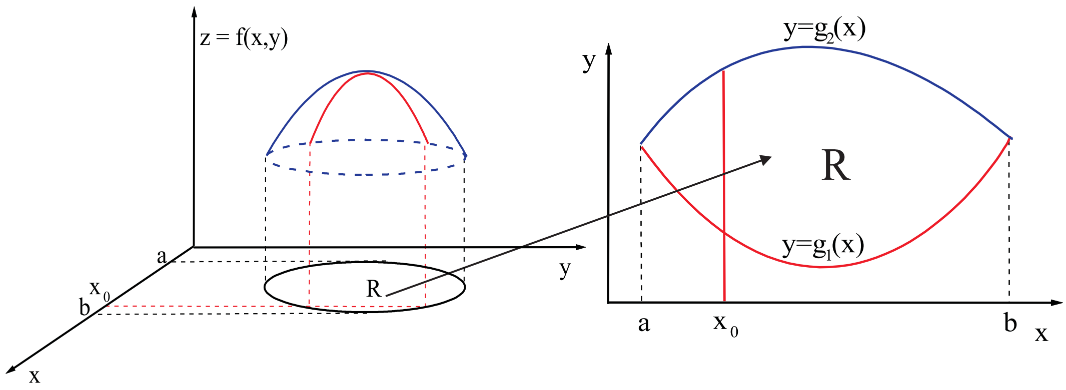


Figure 14: *Computation of the volume enclosed by the function $f(x, y)$. The figure on the r.h.s. shows the region R which is now determined by the intersection of the functions $y = g_1(x)$ and $y = g_2(x)$.*

In such a situation, the volume under the curve $f(x, y)$ is obtained by performing the following integral

$$V = \int_{x=a}^{x=b} dx \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)dy, \tag{2.252}$$

here the first integral

$$\int_{y=g_1(x)}^{y=g_2(x)} f(x, y)dy, \tag{2.253}$$

is an integral in y for x constant and gives us the surface of the region determined by the intersection of the plane $x =$const. and the function $f(x, y)$. This intersection is shown on the l.h.s. of figure 14 in red for a constant value of $x = x_0$. Once we have computed this integral, the integral in x adds up all these areas for all possible values of x giving us the desired volume.

Alternatively, we can consider the same situation parameterizing the integration region as in figure 15. In this case the integral will be given by

$$V = \int_{y=c}^{y=d} dy \int_{x=h_1(y)}^{x=h_2(y)} f(x, y)dx, \tag{2.254}$$

and should of course give the same value as (2.252). The only thing we have done is to change the order of integration, which in this case is a bit more complicated than in the previous examples. The fact that (2.252) and (2.254) are equivalent is important in practise, since it often happens integrating in a certain order is much easier than in the other.

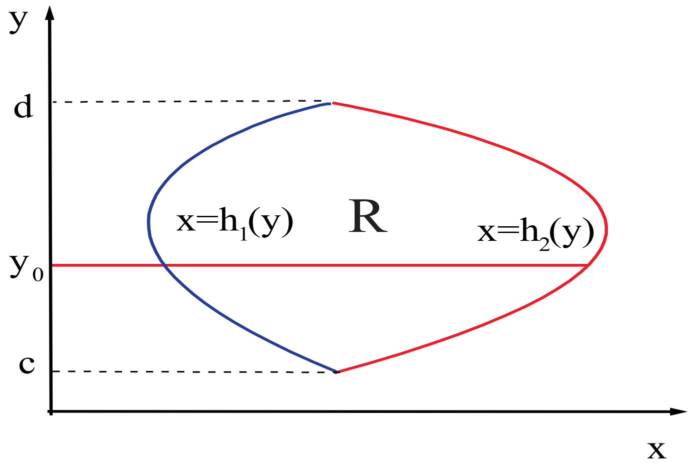


Figure 15: *The integration region R, now determined by the intersection of the curves $x = h_1(y)$ and $x = h_2(y)$.*

Example 1: Compute the integral of the function

$$f(x, y) = yx^2, \tag{2.255}$$

in the region

$$0 \leq x \leq 1, \qquad 0 \leq y \leq x. \tag{2.256}$$

The first thing we must do in this kind of problems is to sketch the integration region. In this case this is quite simple, as we see in the picture below:

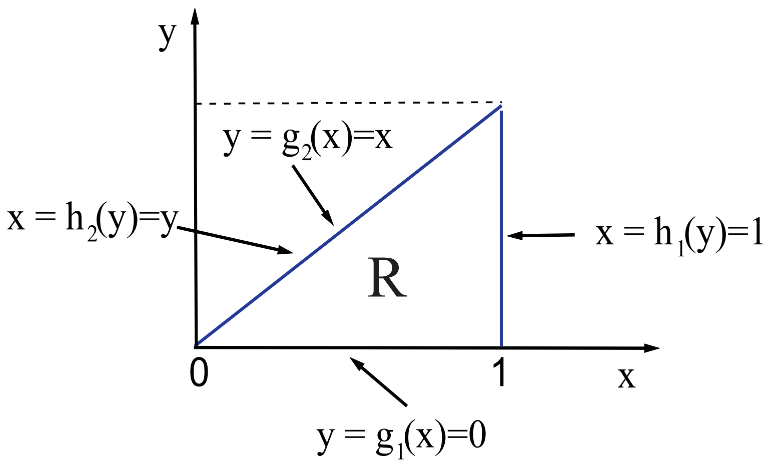


Figure 16: *The integration region for examples 1 and 2.*

Therefore the integral we need to evaluate is

$$V = \int_{x=0}^{x=1} dx \int_{y=0}^{y=x} yx^2 dy. \tag{2.257}$$

The integral in y gives

$$\int_{y=0}^{y=x} yx^2 dy = \left[\frac{y^2 x^2}{2}\right]_0^x = \frac{x^4}{2} - 0 = \frac{x^4}{2}. \tag{2.258}$$

Therefore the volume is

$$V = \int_{x=0}^{x=1} \frac{x^4}{2} dx = \left[\frac{x^5}{10}\right]_0^1 = \frac{1}{10} - 0 = \frac{1}{10}. \tag{2.259}$$

Now we can change the order of integration and check that we get the same result. From the picture above we immediately identify the functions $h_1(y), h_2(y)$ which define the region of integration for the variable x we have

$$V = \int_{y=0}^{y=1} dy \int_{x=y}^{x=1} yx^2 dx. \tag{2.260}$$

The integral in x is

$$\int_{x=y}^{x=1} yx^2 dx = \left[\frac{yx^3}{3}\right]_y^1 = \frac{y}{3} - \frac{y^4}{3} = \frac{y}{3}(1 - y^3). \tag{2.261}$$

Integrating now in y we obtain the final result

$$V = \int_{y=0}^{y=1} \frac{y}{3}(1 - y^3) dy = \left[\frac{y^2}{6} - \frac{y^5}{15}\right]_0^1 = \left(\frac{1}{6} - \frac{1}{15}\right) - 0 = \frac{1}{10}. \tag{2.262}$$

Example 2: Compute the integral

$$I = \int_{y=0}^{y=1} dy \int_{x=y}^{x=1} \cos\left(\frac{\pi x^2}{2}\right) dx, \tag{2.263}$$

by changing the order of integration.
In this case, the integration region is exactly the same as in the previous exercise. However, this is the kind of integral which can be only solved easily when integrating in a particular order. If we try to solve the integral by integrating as in (2.263) we will soon realize that it can not be done (the integral in x is actually a very difficult one). However, we can change the order of integration, as they tell us to do in the problem. To do that, we have to be careful about changing the limits of integration correctly. For that, we only need to look at figure 16 and we find that we can write I equivalently as

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=x} \cos\left(\frac{\pi x^2}{2}\right) dy. \tag{2.264}$$

The integral

$$\int_{y=0}^{y=x} \cos\left(\frac{\pi x^2}{2}\right) dy = \left[y \cos\left(\frac{\pi x^2}{2}\right)\right]_0^x = x \cos\left(\frac{\pi x^2}{2}\right) - 0 = x \cos\left(\frac{\pi x^2}{2}\right), \tag{2.265}$$

is trivial to do, since the argument does not depend on y. Now the second integral is also very easy to do, since we have the product of the cosine of a function and the derivative of that function, therefore

$$I = \int_{x=0}^{x=1} x \cos\left(\frac{\pi x^2}{2}\right) dx = \left[\frac{1}{\pi} \sin\left(\frac{\pi x^2}{2}\right)\right]_0^1 = \frac{1}{\pi} - 0 = \frac{1}{\pi}. \tag{2.266}$$

If you do not realize how to do the integral directly, you can also change variables to $t = \pi x^2/2$ which gives $dt = \pi x dx$ and allows you to rewrite the integral above as

$$I = \int_{x=0}^{x=1} x \cos\left(\frac{\pi x^2}{2}\right) dx = \frac{1}{\pi}\int_{t=0}^{t=\pi/2} \cos(t)\, dt = \frac{1}{\pi}\left[\sin(t)\right]_0^{\pi/2} = \frac{1}{\pi}. \tag{2.267}$$

The conclusion from this exercise is that we must always keep in mind that changing the order of integration might simplify a problem very much.

Example 3: Compute the integral

$$I = \int_{x=0}^{x=1} dx \int_{y=x}^{y=\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy. \tag{2.268}$$

In this case we are not told explicitly to change the order of integration, but we can have a look at the integral and see if that would help. If you look at the integral in y

$$\int_{y=x}^{y=\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy, \tag{2.269}$$

you see that it is not a trivial integral. It can be done, but we need to perform a change of variables of the type $y = x\cosh\theta$. However, if the order of integration could be changed then actually an integral of the type

$$\int \frac{x}{\sqrt{x^2+y^2}} dx, \tag{2.270}$$

is trivial to do, since $x/\sqrt{x^2+y^2}$ is exactly the derivative with respect to x of $\sqrt{x^2+y^2}$. Therefore, changing the order of integration is the best way to solve this problem too. In order to do so, we have first to sketch the integration region in (2.268). This is done in figure 17.

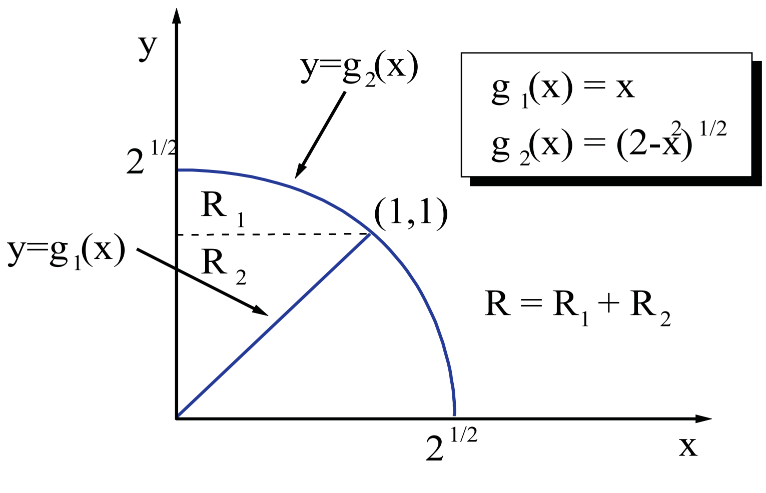


Figure 17: *The integration region for example 3.*

The integration region R is the region between the curves $y = x$ and $y = \sqrt{2-x^2}$. Since in the integral $y = x$ is the lower integration limit, this means that the integral only makes sense in the region where $x \leq y \leq \sqrt{2-x^2}$

$$R := \{(x,y) : 0 \leq x \leq 1, \quad x \leq y \leq \sqrt{2-x^2}\}. \tag{2.271}$$

In the picture that is the region resulting from the union of R_1 and R_2. The form of the curve $y = \sqrt{2-x^2}$ can be easily deduced by taking the square on both sides of the equation. That gives us the equation $y^2 + x^2 = 2$ which is the equation of a disk of radius $\sqrt{2}$. Then we have to choose positive values of x (remember that x is between 0 and 1!) and positive values of y (because of the square root) which gives us just the region $R = R_1 + R_2$ shown in the picture.

Now we need to change the order of integration, this means we need to re-express the integration limits in the variables x and y in such a way that now the integral in x varies between functions of y and the integral in y varies between two constant values. It is not difficult to see that $R = R_1 + R_2$ with

$$R_1 := \{(x,y) : 1 \le y \le \sqrt{2}, \quad 0 \le x \le \sqrt{2-y^2}\}, \tag{2.272}$$

whereas the region R_2 corresponds to

$$R_2 := \{(x,y) : 0 \le y \le 1, \quad 0 \le x \le y\}, \tag{2.273}$$

therefore we can divide our integral in two parts and write $I = I_1 + I_2$ with

$$I_2 = \int_{y=0}^{y=1} dy \int_{x=0}^{x=y} \frac{x\,dx}{\sqrt{x^2+y^2}}, \tag{2.274}$$

$$I_1 = \int_{y=1}^{y=\sqrt{2}} dy \int_{x=0}^{x=\sqrt{2-y^2}} \frac{x\,dx}{\sqrt{x^2+y^2}}. \tag{2.275}$$

These integrals can be now easily computed since:

$$\int_{x=0}^{x=y} \frac{x\,dx}{\sqrt{x^2+y^2}} = \left[\sqrt{x^2+y^2}\right]_0^y = y\sqrt{2} - y = y(\sqrt{2}-1), \tag{2.276}$$

$$\int_{x=0}^{x=\sqrt{2-y^2}} \frac{x\,dx}{\sqrt{x^2+y^2}} = \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} = \sqrt{2} - y. \tag{2.277}$$

Therefore

$$I_2 = \int_{y=0}^{y=1} y(\sqrt{2}-1)dy = \left[\frac{(\sqrt{2}-1)y^2}{2}\right]_0^1 = \frac{\sqrt{2}-1}{2}. \tag{2.278}$$

$$\begin{aligned} I_1 &= \int_{y=1}^{y=\sqrt{2}} (\sqrt{2}-y)dy = \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}} = 2 - 1 - \sqrt{2} + \frac{1}{2} \\ &= \frac{3}{2} - \sqrt{2}. \end{aligned} \tag{2.279}$$

Therefore the value of the integral is

$$I = I_1 + I_2 = (\sqrt{2}-1)2 + \frac{3}{2} - \sqrt{2} = 1 - \frac{\sqrt{2}}{2}. \tag{2.280}$$

Volume from triple integrals: In this section we will learn another method to compute volumes. The typical problem we want to solve is to find the volume of the region enclosed by several surfaces. The best way to understand what this means is to see several examples:

Example 1: Let us consider a plane as the one shown in figure 18.

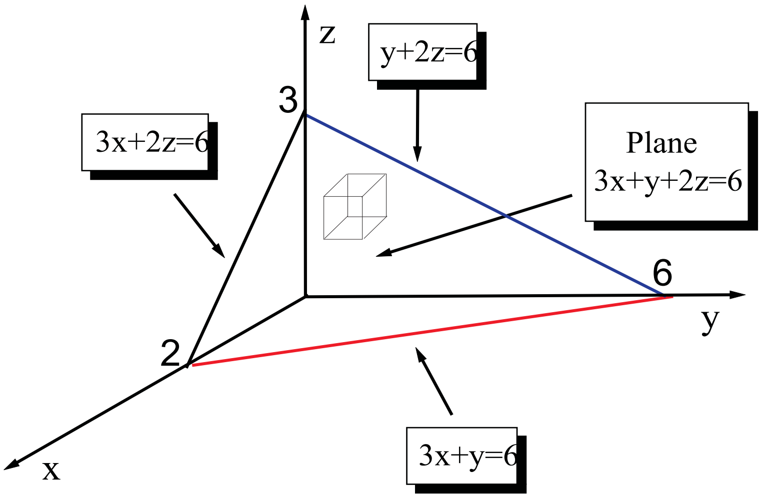


Figure 18: *The integration region for the problem of obtaining the volume enclosed by the plane $3x + y + 2z = 6$ and the coordinate planes $x = 0, y = 0$ and $z = 0$.*

That is the plane $3x + y + 2z = 6$. In the picture we see the three lines corresponding to the intersection of this plane with the three coordinate planes $x = 0, y = 0$ and $z = 0$. How would we compute the volume enclosed by this plane and the three coordinate planes? In this case, we could actually use the same method from the previous section, since we are just trying to obtain the volume below the plane $3x + y + 2z = 6$, that is $z = f(x, y) = 1/2(6 - y - 3x)$. The integration region in the xy-plane can be easily seen from the picture above and we obtain,

$$V = \frac{1}{2}\int_{x=0}^{2} dx \int_{y=0}^{6-3x} (6 - y - 3x)dy, \tag{2.281}$$

which could compute as before. However, we want to solve this problem in an slightly different (but equivalent) way.

We notice that the volume of an infinitesimal cube inside the region whose volume we want to compute (like the one in the figure 18) would be given by

$$dV = dxdydz, \tag{2.282}$$

therefore, the total volume would be

$$V = \int\int\int_R dxdydz, \tag{2.283}$$

and the main difficulty of the problem is to determine R, the integration region. This is however quite easy from the picture above

$$R = \{(x, y, z) : 0 \leq x \leq 2, \quad 0 \leq y \leq 6 - 3x, \quad 0 \leq z \leq \frac{6 - 3x - y}{2}\}, \tag{2.284}$$

therefore

$$V = \int_{x=0}^{x=2} dx \int_{y=0}^{y=6-3x} dy \int_{z=0}^{z=(6-3x-y)/2} dz, \tag{2.285}$$

the integral in z is

$$\int_{z=0}^{z=(6-3x-y)/2} dz = [z]_0^{(6-3x-y)/2} = \frac{6-3x-y}{2}, \tag{2.286}$$

now we have to plug this result into the integral in y,

$$\begin{aligned} \int_{y=0}^{y=6-3x} \frac{6-3x-y}{2} dy &= \left[\frac{(6-3x)y}{2} - \frac{y^2}{4}\right]_0^{6-3x} \\ &= \frac{(6-3x)(6-3x)}{2} - \frac{(6-3x)^2}{4} = \frac{(6-3x)^2}{4}, \end{aligned} \tag{2.287}$$

and finally we plug this result into the last integral to obtain

$$\begin{aligned} V &= \int_{x=0}^{x=2} \frac{(6-3x)^2}{4} dx = \int_{x=0}^{x=2} \frac{36+9x^2-36x}{4} dx \\ &= \left[9x + \frac{3x^3}{4} - \frac{9x^2}{2}\right]_0^2 = 18 + 6 - 18 = 6. \end{aligned} \tag{2.288}$$

Notice that, once we have done the integral in z, we recover the integral (2.281).

<u>**Note:**</u> The main difficulty of this kind of problems is finding the integration region. You must always start trying to get an idea of what it looks like. It might be enough to look at the projection of the integration region onto the xy-plane.

Example 2: Find the volume of the solid bounded by the following surfaces:

$$\begin{aligned} z &= x+2, && \text{plane perpendicular to the } y=0 \text{ plane}, \\ z &= 0, && \text{the coordinate plane } z=0, \\ y &= x^2, && \text{one of the quadratic surfaces, a parabolic cylinder}, \\ y &= 2x+3, && \text{plane perpendicular to the } z=0 \text{ plane}. \end{aligned} \tag{2.289}$$

In this case there are several surfaces involved and therefore the volume is not just the volume under a certain surface. Here we have to use the method that we just learned. The first step is to sketch the solid bounded by these surfaces. We are told in the problem that all the equations above describe surfaces. This is a very important information, since if they will not tell us anything, then we would say that $z = x+2$ is a line in the xz-plane, $y = x^2$ is a parabola in the xy-plane and $y = 2x+3$ is a line in the xy-plane. But, since we are told that these equations describe surfaces, then we must assume that for example $z = x+2$ is an equation which holds **for all values of** y. Therefore this surface is generated by drawing the line $z = x+2$ and then shifting this line along the y direction. The same strategy holds for the other surfaces. Doing that we should obtain a picture like the one below.

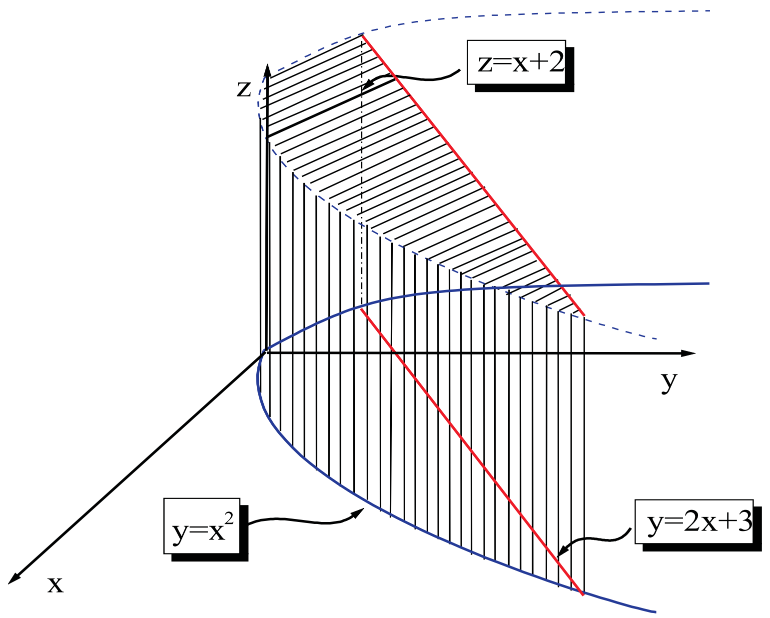
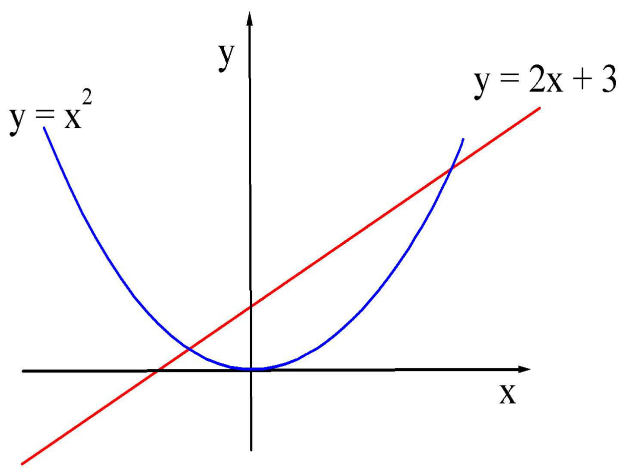


Figure 19: *The integration region for the problem of obtaining the volume enclosed by the planes $z = 0$, $z = x + 2$ and $y = 2x + 3$ and the surface $y = x^2$.*

From the equations of the surfaces (2.289) we have that the integration regions for the variables z and y are

$$0 \leq z \leq x + 2, \qquad x^2 \leq y \leq 2x + 3. \tag{2.290}$$

Notice that $x^2 \leq y \leq 2x + 3$ and not $2x + 3 \leq y \leq x^2$. This follows from figure 19, where we can see that the region of integration corresponds always to $x^2 \leq 3x + 2$. This is even easier to see if we look at the projection of the integration region onto the xy-plane,



Finally, we only need to determine the integration region for x, that is the maximum and minimum values x can take in the region of figure 19. These values correspond in fact to the solutions of the equation

$$x^2 - 2x - 3 = (x + 1)(x - 3) = 0 \quad \Rightarrow \quad x = -1, 3, \tag{2.291}$$

that is the meeting points of the line $y = 2x + 3$ and the parabola $y = x^2$. Therefore, the integration region is

$$R = \{(x, y, z) : -1 \leq x \leq 3, \quad x^2 \leq y \leq 2x + 3, \quad 0 \leq z \leq x + 2\}. \tag{2.292}$$

We can now obtain the volume of the solid,

$$V = \int_{x=-1}^{x=3} dx \int_{y=x^2}^{y=2x+3} dy \int_{z=0}^{z=x+2} dz. \tag{2.293}$$

The first integral gives

$$\int_{z=0}^{z=x+2} dz = [z]_0^{x+2} = x + 2. \tag{2.294}$$

Substituting this result into the initial integral we compute now the integral in the variable y,

$$\begin{aligned} \int_{y=x^2}^{y=2x+3} (x+2)dy &= (x+2)\,[y]_{x^2}^{2x+3} = (x+2)(2x+3-x^2) \\ &= -x^3 + 7x + 6. \end{aligned} \tag{2.295}$$

Finally, we only have to substitute the previous result into the last integral to obtain the volume

$$\begin{aligned} V &= \int_{x=-1}^{x=3} (-x^3 + 7x + 6)dx = \left[-\frac{x^4}{4} + \frac{7x^2}{2} + 6x\right]_{-1}^{3} \\ &= \left(-\frac{81}{4} + \frac{63}{2} + 18\right) - \left(-\frac{1}{4} + \frac{7}{2} - 6\right) = 32. \end{aligned} \tag{2.296}$$

Example 3: The region R in the positive octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) is bounded by the surface $y = 4x^2$ and by the planes $x = 0, y = 4, z = 0$ and $z = 2$. Evaluate the volume integral

$$I = \int\int\int_R 2x dx dy dz. \tag{2.297}$$

As usual, let us sketch the integration region

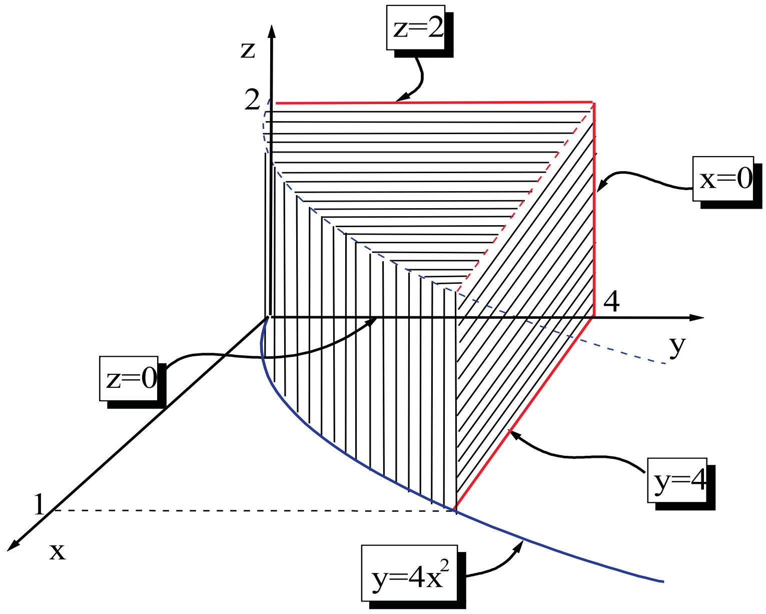


Figure 20: *The integration region of example 3, that is the region enclosed by the planes $x = 0, z = 0, z = 2, y = 4$ and the surface $y = 4x^2$.*

From the picture and the information given by the problem we can deduce that the integration region is

$$R = \{(x, y, z) : 0 \leq x \leq 1, \quad 4x^2 \leq y \leq 4, \quad 0 \leq z \leq 2.\} \tag{2.298}$$

The integral is

$$I = 2 \int_{x=0}^{x=1} x dx \int_{y=4x^2}^{4} dy \int_{z=0}^{2} dz. \tag{2.299}$$

The integral in z is

$$\int_{z=0}^{2} dz = [z]_0^2 = 2. \tag{2.300}$$

The integral in y is

$$\int_{y=4x^2}^{4} dy = [y]_{4x^2}^4 = 4(1 - x^2). \tag{2.301}$$

Therefore, the final result is

$$I = 16 \int_{x=0}^{x=1} x(1 - x^2) dx = 16 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = 16 \left(\frac{1}{2} - \frac{1}{4} \right) = 4. \tag{2.302}$$

2.5.2 Standard coordinate systems in $\mathbb{R}^2$ and $\mathbb{R}^3$

Similarly as for functions of one variable, integrals of functions of two or three variables may become simpler when changing coordinates in an appropriate way. For this reason we are going to introduce in this section the so-called standard coordinate systems which are used in 2 and 3-dimensions. Knowing these coordinate systems will allow us in the next section to perform changes of variables in volume integrals.

Coordinate systems in $\mathbb{R}^2$

There are two standard coordinate systems which are used to describe points in 2-dimensional space. These coordinate systems are

- the **Cartesian coordinate system** (which we normally use), in which we characterize points by two coordinates (x, y) and
- the **Polar coordinate system** in which we characterize points in the 2-dimensional xy-plane by their distance to the origin r and the angle θ (see figure 21).

As you see in the picture, both coordinate systems are related by the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{2.303}$$

with $0 \leq \theta \leq 2\pi$, or equivalently

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{2.304}$$

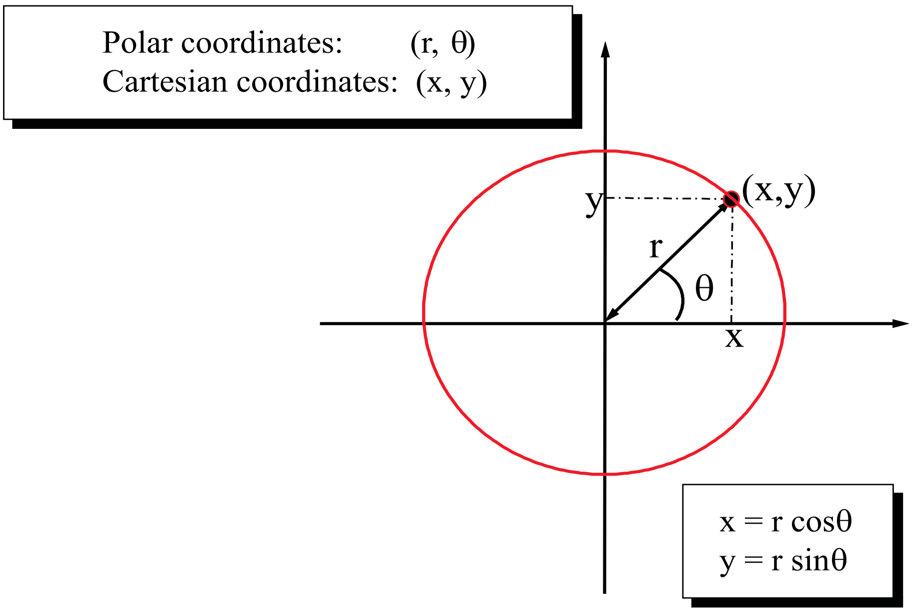


Figure 21: *The polar and Cartesian coordinate systems.*

Coordinate systems in $\mathbb{R}^3$

There are three standard coordinate systems which are used to describe points in 3-dimensional space. These coordinate systems are

- the **Cartesian coordinate system** (which we normally use), in which we characterize points by three coordinates (x, y, z) and
- the **cylindrical coordinate system**: this coordinate system is a sort of generalization of polar coordinates in two dimensions. In cylindrical coordinates a point in 3-dimensional space is characterized by coordinates (r, θ, z), which are defined as shown in figure 22 (they are the same as in polar coordinates plus one extra coordinate describing the height in the z direction),
- the **spherical coordinate system**, in which a point in 3-dimensional space is characterized by the distance to the origin r and the angles θ, ϕ defined in figure 23,

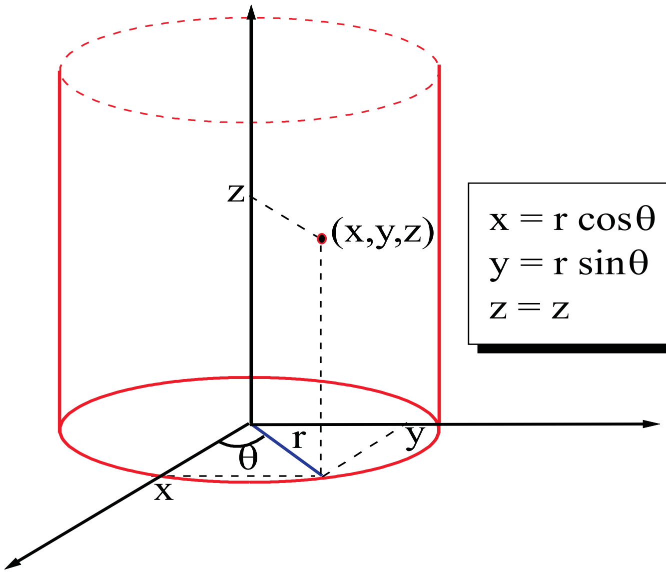


Figure 22: *The cylindrical and Cartesian coordinate systems.*

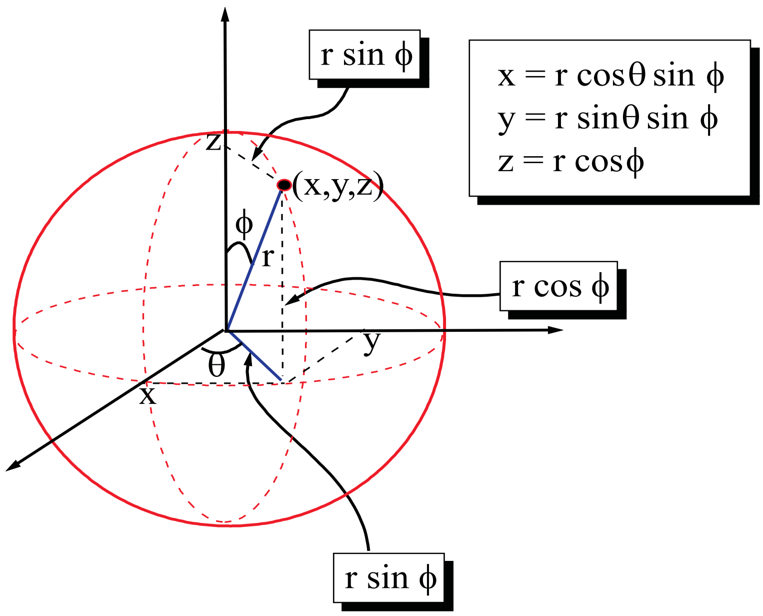


Figure 23: *The spherical and Cartesian coordinate systems.*

The relation between cylindrical and spherical coordinates and Cartesian coordinates is given in the figures 21 and 23. For cylindrical coordinates we have

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \tag{2.305}$$

with $0 \leq \theta \leq 2\pi$ or equivalently

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad z = z. \tag{2.306}$$

For spherical coordinates we have

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi, \tag{2.307}$$

with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$ or equivalently

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad \phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right). \tag{2.308}$$

<u>**Note:**</u> **When is it convenient to use polar, cylindrical or spherical coordinates instead of Cartesian coordinates?** Essentially it depends on the characteristics of the integration region they are asking us to consider. For example, suppose we are asked to solve the following problem: **compute the volume of a sphere of radius 3**, characterized by the equation

$$x^2 + y^2 + z^2 = 9. \tag{2.309}$$

We can try to solve the problem in the same way we have seen in the previous examples. We must sketch the region of integration (see figure 23), then determine the integration region in x, y and z and compute the integral

$$\int\int\int_R dxdydz. \tag{2.310}$$

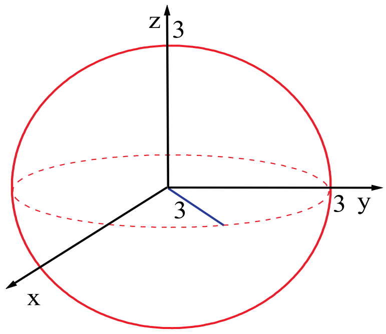


Figure 24: *The integration region of our problem.*

In terms of the coordinates x, y and z, the integration region is relatively complicated. In fact it is given by

$$\begin{aligned} R = \Big\{(x, y, z) : -3 \leq x \leq 3, \quad -\sqrt{9 - x^2} \leq y \leq \sqrt{9 - x^2}, \\ -\sqrt{9 - x^2 - y^2} \leq z \leq \sqrt{9 - x^2 - y^2}\Big\}. \end{aligned} \tag{2.311}$$

If we now try to compute the integral we will soon see that it becomes rather complicated

$$V=\int_{x=-3}^{x=3}dx\int_{y=-\sqrt{9-x^2}}^{x=\sqrt{9-x^2}}dy\int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}}dz. \tag{2.312}$$

The first integral gives

$$\int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}}dz=[z]_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}}=2\sqrt{9-x^2-y^2}. \tag{2.313}$$

Inserting this result into the integral in y we have

$$\int_{y=-\sqrt{9-x^2}}^{x=\sqrt{9-x^2}}2\sqrt{9-x^2-y^2}dy. \tag{2.314}$$

This integral is not completely trivial to do. The best way to do it is to change coordinates as

$$y=\sqrt{9-x^2}\cos\alpha \quad\Rightarrow\quad dy=-\sqrt{9-x^2}\sin\alpha d\alpha, \tag{2.315}$$

with $\pi\leq\alpha\leq 0$. Changing coordinates that way, the integral (2.314) becomes

$$\begin{aligned}-\int_{\alpha=\pi}^{\alpha=0}2(9-x^2)\sin^2(\alpha)d\alpha &= \int_{\alpha=0}^{\alpha=\pi}(9-x^2)(1-\cos(2\alpha))d\alpha\\ &= (9-x^2)\left[\alpha-\frac{\sin(2\alpha)}{2}\right]_0^{\pi}=(9-x^2)\pi. \end{aligned}\tag{2.316}$$

Finally, integrating in x we obtain the volume

$$\begin{aligned}V &= \pi\int_{x=-3}^{x=3}(9-x^2)dx=\pi\left[9x-\frac{x^3}{3}\right]_{-3}^{3}\\ &= \pi\left(27-\frac{27}{3}\right)-\pi\left(-27+\frac{27}{3}\right)=36\pi. \end{aligned}\tag{2.317}$$

Therefore, the final result is 36π which is indeed the volume of a sphere of radius 3[‡]. This way of computing the volume is correct but it is in fact much more complicated than if we had used spherical coordinates from the beginning. In that case, the integration region is very easy to determine

$$R=\{(x,y,z):0\leq r\leq 3,\quad 0\leq\theta\leq 2\pi,\quad 0\leq\phi\leq\pi\}, \tag{2.318}$$

and **the only difficulty is to determine how $dx\,dy\,dz$ can be expressed in terms of $dr\,d\theta\,d\phi$**. We will see in the next section how this relation can be found. The result we are going to find is

$$dx\,dy\,dz=r^2\sin\phi\,dr\,d\theta\,d\phi. \tag{2.319}$$

[‡]Remember that the volume of a sphere of radius r is given by:

$$V=\frac{4}{3}\pi r^3.$$

If we know this, then we can compute our integral very easily. It is just

$$\begin{aligned} V &= \int_{r=0}^{r=3} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi \\ &= \int_{r=0}^{r=3} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \, [-\cos\phi]_0^{\pi} = 2 \int_{r=0}^{r=3} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \\ &= 2 \int_{r=0}^{r=3} r^2 dr \, [\theta]_0^{2\pi} = 4\pi \int_{r=0}^{r=3} r^2 dr = 4\pi \left[\frac{r^3}{3}\right]_0^3 = 4\pi \frac{27}{3} = 36\pi. \end{aligned} \tag{2.320}$$

We can say as a conclusion that whenever the integration region is a sphere (or part of a sphere), we must use spherical coordinates. If the integration region is a cylinder (or part of a cylinder), we must use cylindrical coordinates. If the integration region is a disk (or part of one) it is best to use polar coordinates.

2.5.3 Change of variables and Jacobians

In the previous example we saw that, once we have identified the type of coordinates which is best to use for solving a particular problem, the next step is to do the change of coordinates on the integral we want to compute. One way to see how this goes, is to draw a picture of an infinitesimal element of volume (or surface, if we are doing an integral of a function of two variables) and compute its volume (surface) in terms of the new variables. Let us do that for the simplest case of two variables.

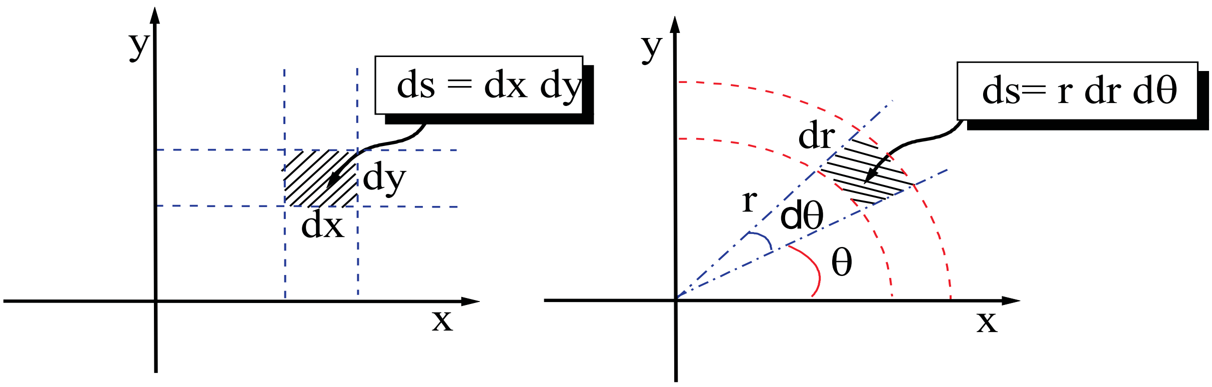


Figure 25: *Differentials of surface.*

Consider an infinitesimal rectangle in Cartesian coordinates. Its area is given by $ds = dx\,dy$. What is the surface of an elementary infinitesimal region in polar coordinates? The answer follows from figure 25, that is

$$dx\,dy = r\,d\theta\,dr. \tag{2.321}$$

Therefore, given an integral

$$I = \int\int_R f(x,y)dxdy, \tag{2.322}$$

a change to polar coordinates will give

$$I = \int\int_{R'} f(r\cos\theta, r\sin\theta)rd\theta dr, \tag{2.323}$$

where R' is the integration region R in terms of the new coordinates.

Example: Compute the integral

$$I = \int\int_R \sqrt{x^2+y^2}dxdy, \tag{2.324}$$

on a disk of radius a. This is a typical case in which the best is to use polar coordinates (the integration region is a disk!). In polar coordinates

$$R = \{(r,\theta) : 0 \leq r \leq a, \qquad 0 \leq \theta \leq 2\pi\}. \tag{2.325}$$

In addition, we have seen before that

$$\sqrt{x^2+y^2} = r, \qquad dx\,dy = r\,dr\,d\theta, \tag{2.326}$$

therefore, the integral in polar coordinates is simply

$$I = \int_{r=0}^{r=a} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta. \tag{2.327}$$

The integral in θ is just

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi. \tag{2.328}$$

So we finally get

$$I = 2\pi \int_{r=0}^{r=a} r^2 dr = 2\pi \left[\frac{r^3}{3}\right]_0^a = \frac{2\pi a^3}{3}. \tag{2.329}$$

We have found the result (2.321) from geometrical considerations (from figure 25). This can be done for any change of coordinates, in 2 or 3 dimensions. However there is a more systematic way to compute the element of volume or surface under a change of coordinates. In general we have:

Definition: Let (x, y) be the Cartesian coordinates in 2-dimensional space and consider a generic change of variables

$$x = x(u, v), \qquad \text{and} \qquad y = y(u, v), \tag{2.330}$$

(u, v) being the new variables. Then the differentials of surface are related in the following way

$$dx\, dy = |J|\, du\, dv, \tag{2.331}$$

where $|J|$ is the **modulus** of the following determinant

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \tag{2.332}$$

This determinant is called **the Jacobian of the transformation of coordinates**.

Example 1: Use the Jacobian to obtain the relation between the differentials of surface in Cartesian and polar coordinates.

The relation between Cartesian and polar coordinates was given in (2.303). We can easily compute the Jacobian,

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r. \tag{2.333}$$

Therefore

$$dx\, dy = r\, dr\, d\theta, \tag{2.334}$$

which is the same result as (2.321).

Example 2: Find the Jacobian of the transformation

$$x = v/u, \qquad \text{and} \qquad y = v. \tag{2.335}$$

Using these new variables evaluate the integral

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{y^2}{x^2} e^{y/x} dx dy \tag{2.336}$$

We start by computing the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -v/u^2 & 1/u \\ 0 & 1 \end{vmatrix} = -\frac{v}{u^2}. \tag{2.337}$$

Therefore

$$dx\, dy = |J| du\, dv = \frac{v}{u^2} du\, dv. \tag{2.338}$$

Now we have to transform the function we want to integrate,

$$\frac{y^2}{x^2} e^{y/x} = u^2 e^u, \tag{2.339}$$

and we have to find the new integration region

$$0 \le x \le 1 \quad \Leftrightarrow \quad 0 \le v \le u, \tag{2.340}$$

$$0 \le y \le x \quad \Leftrightarrow \quad 0 \le u \le 1. \tag{2.341}$$

Therefore the integral we need to compute is

$$I = \int_{u=0}^{u=1} e^u du \int_{v=0}^{v=u} v\, dv. \tag{2.342}$$

The first integral is

$$\int_{v=0}^{v=u} v\, dv = \left[\frac{v^2}{2} \right]_{v=0}^{v=u} = \frac{u^2}{2}, \tag{2.343}$$

and so

$$I = \frac{1}{2} \int_{u=0}^{u=1} u^2 e^u du. \tag{2.344}$$

This integral can be done by using integration by parts twice

$$\begin{aligned} \int_{u=0}^{u=1} u^2 e^u du &= \left[u^2 e^u\right]_0^1 - \int_{u=0}^{u=1} 2ue^u du = e - \int_{u=0}^{u=1} 2ue^u du \\ &= e - [2ue^u]_0^1 + \int_{u=0}^{u=1} 2e^u du = e - 2e + \int_{u=0}^{u=1} 2e^u du \\ &= [2e^u]_0^1 - e = 2e - 2 - e = e - 2. \end{aligned} \tag{2.345}$$

Therefore

$$I = \frac{e-2}{2}. \tag{2.346}$$

Definition: Let (x, y, z) be the Cartesian coordinates in 3-dimensional space and consider a generic change of variables

$$x = x(u, v, t), \quad y = y(u, v, t) \quad \text{and} \qquad z = z(u, v, t), \tag{2.347}$$

(u, v, t) being the new variables. Then the differentials of surface are related in the following way

$$dx\, dy\, dz = |J|\, du\, dv\, dt, \tag{2.348}$$

where $|J|$ is the **modulus** of the following determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{vmatrix} \tag{2.349}$$

This determinant is called **the Jacobian of the transformation of coordinates**.

Example 1: The Jacobian of cylindrical coordinates.

The relation between Cartesian and cylindrical coordinates was given in (2.305). We can easily compute the Jacobian,

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r. \tag{2.350}$$

Therefore

$$dx\, dy\, dz = |J|\, dr\, d\theta\, dz, = r\, dr\, d\theta\, dz, \tag{2.351}$$

Example 2: The Jacobian of spherical coordinates.

The relation between Cartesian and spherical coordinates was given in (2.307). The Jacobian is,

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} \\ &= -r^2\cos^2\theta\sin^3\phi - r^2\sin^2\theta\cos^2\phi\sin\phi - r^2\cos^2\theta\cos^2\phi\sin\phi \\ &- r^2\sin^2\theta\sin^3\phi = -r^2\sin^3\phi(\cos^2\theta + \sin^2\theta) - r^2\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta) \\ &= -r^2\sin^3\phi - r^2\cos^2\phi\sin\phi = -r^2\sin\phi(\sin^2\phi + \cos^2\phi) = -r^2\sin\phi. \end{aligned} \tag{2.352}$$

Therefore

$$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi = r^2 \sin\phi\,dr\,d\theta\,d\phi, \tag{2.353}$$

Let us now see a couple of examples of integral where we use cylindrical and spherical coordinates:

Example 1: Use cylindrical coordinates to evaluate the following integral

$$\int\int\int_R (x^2+y^2)dxdydz, \tag{2.354}$$

where R is the solid bounded by the surface $x^2+y^2=2z$ and the plane $z=2$.

As usual, we start by sketching the integration region. The first equation $x^2+y^2=2z$ is a paraboloid (one of the quadratic surfaces we saw some time ago!). The integration region looks more or less like that

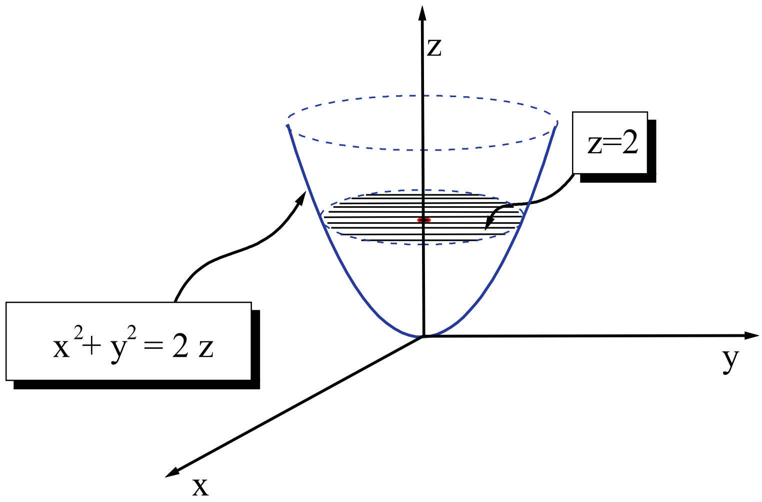


Figure 26: *The integration region of our problem.*

Now we want to do the integral in cylindrical coordinates. We have seen before that

$$x^2+y^2=r^2, \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz \tag{2.355}$$

so the integral we want to compute is

$$\int\int\int_{R'} r^3 dr d\theta dz, \tag{2.356}$$

where R' is the integration region in cylindrical coordinates. From the picture and the information given by the problem it is easy to find

$$R' = \{(r,\theta,z) : 0 \le r \le \sqrt{2z}, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 2\}. \tag{2.357}$$

Therefore our integral is

$$\int_{z=0}^{z=2} dz \int_{\theta=0}^{\theta=2\pi} d\theta \int_{r=0}^{r=\sqrt{2z}} r^3 dr. \tag{2.358}$$

The integral in r is

$$\int_{r=0}^{r=\sqrt{2z}} r^3 dr = \left[\frac{r^4}{4}\right]_0^{\sqrt{2z}} = z^2. \tag{2.359}$$

The integral in θ is simply

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi. \tag{2.360}$$

Therefore, the final result is

$$2\pi \int_{z=0}^{z=2} z^2 dz = \left[\frac{z^3}{3}\right]_0^2 = \frac{16\pi}{3}. \tag{2.361}$$

Example 2: Use spherical coordinates to compute the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.

The region of integration for this problem is given in the picture below:

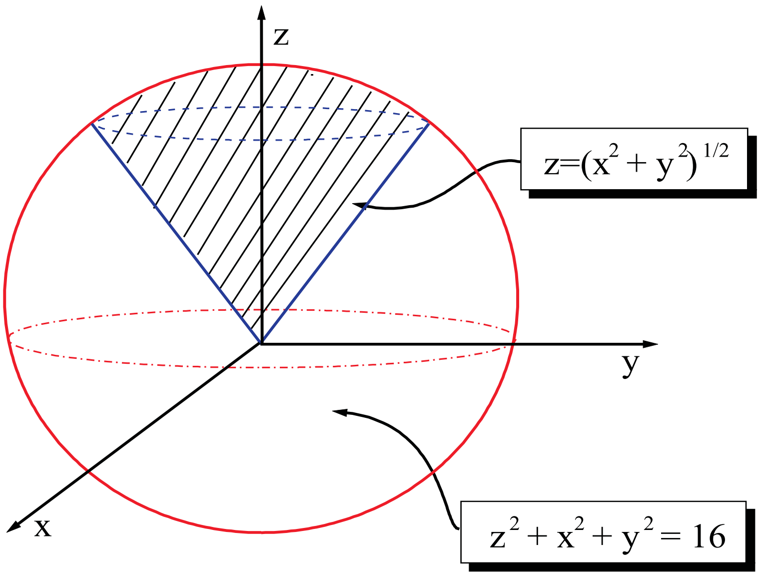


Figure 27: *The sphere $x^2 + y^2 + z^2 = 16$ and the cone $z = \sqrt{x^2 + y^2}$. The dashed region is our integration region.*

The integral we have to compute is

$$\int\int\int_R dxdydz = \int\int\int_{R'} r^2 \sin\phi dr d\theta d\phi. \tag{2.362}$$

Here we have only used the result (2.353) and we called R' the integration in spherical coordinates. In order to determine R' we notice the following: the equation of the sphere in figure 26 in spherical coordinates (see (2.307)) is just

$$x^2 + y^2 + z^2 = r^2 = 16, \tag{2.363}$$

and the equation of the cone is

$$r\cos\phi = \sqrt{r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta} = r\sin\phi, \tag{2.364}$$

from this equation it follows

$$\tan\phi = 1 \quad \Rightarrow \quad \phi = \frac{\pi}{4}. \tag{2.365}$$

This is the angle of the cone with respect to the z axes. Therefore, we just have to integrate in the following region

$$R' = \{(r,\theta,\phi) : 0 \le r \le 4, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/4\}. \tag{2.366}$$

The volume is thus

$$V = \int_{r=0}^{r=4} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi/4} \sin\phi\, d\phi, \tag{2.367}$$

with

$$\int_{\phi=0}^{\phi=\pi/4} \sin\phi\, d\phi = [-\cos\phi]_{\phi=0}^{\phi=\pi/4} = -\frac{1}{\sqrt{2}} + 1 = \frac{2-\sqrt{2}}{2}. \tag{2.368}$$

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi, \tag{2.369}$$

and

$$\int_{r=0}^{r=4} r^2 dr = \left[\frac{r^3}{3}\right]_0^4 = \frac{64}{3}. \tag{2.370}$$

Therefore the volume is

$$V = \frac{64\pi(2-\sqrt{2})}{3}. \tag{2.371}$$

3 Differential equations

In this last part of the Calculus course we are going to study some new methods to solve certain types of differential equations. This will be a continuation of what you studied in your 1st year calculus course and, as in there, we are going to deal exclusively with **real functions** $y = f(x)$ **of one real variable**. As last year, we will concentrate on a subclass of differential equations, that is **linear differential equations**.

3.1 Linear differential equations

<u>**Definition:**</u> A **n-th order linear differential equation** is an equation of the form

$$y^{(n)} + P_1(x)y^{(n-1)} + P_2(x)y^{(n-2)} + \cdots + P_n(x)y = R(x), \tag{3.1}$$

where

$$y^{(k)} = \frac{d^k y}{dx^k}, \tag{3.2}$$

is the k-th derivative of the function y with respect to the variable x and $P_1(x), \ldots P_n(x)$ and $R(x)$ are real functions of x which are continuous in a certain interval $I \in \mathbb{R}$. The **order** of the differential equation is the highest order of the derivatives appearing in it (in this case n). The equation is **linear** because no terms involving products of y and its derivatives appear (e.g. terms like $yy^{(k)}$ or $y^{(p)}y^{(k)}$).

<u>**Notation:**</u> In order to prove certain general properties of linear differential equations it is convenient to introduce various notations. For example, we can introduce an operator L which acts on a function $f(x)$ as follows

$$L(f) = f^{(n)} + P_1(x)f^{(n-1)} + P_2(x)f^{(n-2)} + \cdots + P_n(x)f, \tag{3.3}$$

that is,

$$L = D^n + P_1(x)D^{n-1} + P_2(x)D^{n-2} + \cdots + P_n(x), \tag{3.4}$$

with

$$D = \frac{d}{dx} \qquad \text{and} \qquad D^n = \frac{d^n}{dx^n}. \tag{3.5}$$

In terms of the operator L, the equation (3.1) would take the form

$$L(y) = R(x) = (D^n + P_1(x)D^{n-1} + P_2(x)D^{n-2} + \cdots + P_n(x))y. \tag{3.6}$$

Using the definition of the operator L we can easily deduce the two following properties:

$$L(y_1 + y_2) = L(y_1) + L(y_2) \qquad \text{and} \qquad L(\alpha y) = \alpha L(y), \tag{3.7}$$

which follow from the properties of the derivative. These two properties are equivalent to saying that L is a **linear operator**.

3.1.1 Second order linear differential equations

Let us now consider a particular case of the equations (3.1), that is 2nd order differential equations

$$y'' + P_1(x)y' + P_2(x)y = R(x) = L(y). \tag{3.8}$$

As usual in this context, solving these equations is done in two steps: fist we must solve the homogeneous 2nd-order differential equation

$$y'' + P_1(x)y' + P_2(x)y = 0 = L(y), \tag{3.9}$$

and then find a particular solution of the inhomogeneous equation (3.8). In the next sections we are going to see why this is the case and how to solve (3.8)-(3.9).

Homogeneous equations: In this section we are going to study how to solve homogeneous equations such as (3.9). We will also see two theorems which will answer two fundamental questions: when do solutions to (3.9) exist? and under which conditions is the solution of (3.9) unique? Before entering these details, let us look at a very simple example:

Example: Consider the following homogeneous 2nd order differential equation

$$y'' + k^2 y = 0, \tag{3.10}$$

where $k \neq 0$ is a constant. This is an equations which you have learnt how to solve last year. In general you try solutions of the type:

$$y = Ce^{mx}, \tag{3.11}$$

with C and m being constants. Then you plug this solution into the equation above and obtain

$$m^2 + k^2 = 0 \qquad \Rightarrow \qquad m = \pm ik. \tag{3.12}$$

Therefore the most general solution of (3.10) is of the form

$$y = C_1 e^{ikx} + C_2 e^{-ikx}, \tag{3.13}$$

with C_1, C_2 being generic constants. Equivalently we can write

$$y = A_1 \sin(kx) + A_2 \cos(kx), \tag{3.14}$$

with $C_1 = (A_2 - iA_1)/2$ and $C_2 = (A_2 + iA_1)/2$.

The solution (3.1.1) with A_1, A_2 arbitrary is the most general solution of (3.10). However, in many problems we have to select a particular solution from the set (3.1.1), a solution satisfying certain additional conditions. There conditions are called **initial conditions**. For example, suppose we want to find a solution of (3.10) such that for a certain value of $x = x_0$,

$$y'(x_0) = b \qquad \text{and} \qquad y(x_0) = a. \tag{3.15}$$

This means that the following two equations have to be satisfied

$$\begin{aligned} y(x_0) &= a = A_1 \sin(kx_0) + A_2 \cos(kx_0), & (3.16) \\ y'(x_0) &= b = A_1 k \cos(kx_0) - A_2 k \sin(kx_0). & (3.17) \end{aligned}$$

This is a system of two linear equations for two unknowns. An equivalent way of writing these equations is

$$\begin{pmatrix} \sin(kx_0) & \cos(kx_0) \\ k\cos(kx_0) & -k\sin(kx_0) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.18}$$

A system of equations like this can **always** be solved (we have 2 equations and 2 unknowns). It will have a **unique solution** provided the two equations (3.17) are independent from each other. Another way of saying that is to say that the determinant

$$\begin{vmatrix} \sin(kx_0) & \cos(kx_0) \\ k\cos(kx_0) & -k\sin(kx_0) \end{vmatrix} = k(\cos^2(kx_0) + \sin^2(kx_0)) = k, \tag{3.19}$$

must be non-vanishing ($k \neq 0$). Since it was assumed from the beginning that $k \neq 0$, we can conclude that there is a unique solution of the initial value problem

$$y'' + k^2 y = 0 \quad \text{with} \quad y'(x_0) = b \quad \text{and} \quad y(x_0) = a. \tag{3.20}$$

Existence theorem: Let $P_1(x), P_2(x)$ be continuous functions of x in an open interval I and let

$$L(y) = y'' + P_1(x)y' + P_2(x)y. \tag{3.21}$$

If $x_0 \in I$ and a, b are given real numbers, there exists $y = f(x)$ such that $L(y) = 0$ on I with $f(x_0) = a$ and $f'(x_0) = b$.

Uniqueness theorem: Let

$$L(y) = y'' + P_1(x)y' + P_2(x)y, \tag{3.22}$$

and let $f(x), g(x)$ be solutions of the homogenous equation $L(f) = L(g) = 0$ on an open interval I of $\mathbb{R}$. Assume that $f(x_0) = g(x_0)$ and $f'(x_0) = g'(x_0)$ for some $x_0 \in I$. Then $f(x) = g(x)$ for all $x \in I$.

Theorem (characterization of solutions): Let

$$L(y) = 0 = y'' + P_1(x)y' + P_2(x)y, \tag{3.23}$$

be a 2nd order linear and homogeneous differential equation with coefficients $P_1(x)$, $P_2(x)$ which are continuous in an open interval $I \in \mathbb{R}$. Let $u_1(x), u_2(x)$ be two non-zero functions satisfying $L(u_1) = L(u_2) = 0$ in I and such that $u_2(x)/u_1(x)$ **is not a constant in** I. Then

$$y = c_1 u_1(x) + c_2 u_2(x), \tag{3.24}$$

is a solution of $L(y) = 0$ on I. Conversely, if y is a solution of (3.23) in I, then there exist constants c_1, c_2 such that (3.24) holds. This means that all solutions of (3.23) are of the form (3.24).

Proof:

(a) To proof that $c_1u_1(x)+c_2u_2(x)$ is a solution of (3.23) provided that $u_1(x), u_2(x)$ are solutions of (3.23) we only have to use the first property in (3.7), that is the fact that L is a linear operator. Therefore

$$L(c_1u_1(x) + c_2u_2(x)) = c_1L(u_1(x)) + c_2L(u_2(x)). \tag{3.25}$$

(b) The second statement we have to proof is that all solutions of (3.23) are of the form $y = c_1u_1(x) + c_2u_2(x)$. To prove that we can consider a solution of the equation $y = f(x)$ and choose a point $x_0 \in I$ with initial conditions $y'(x_0) = f'(x_0)$ and $y(x_0) = f(x_0)$. If we are able to show that we can find two constants c_1, c_2 such that

$$\begin{aligned} f(x_0) &= c_1u_1(x_0) + c_2u_2(x_0), && (3.26)\\ f'(x_0) &= c_1u_1'(x_0) + c_2u_2'(x_0), && (3.27) \end{aligned}$$

then we would have proven that f and $c_1u_1(x) + c_2u_2(x)$ have the same values and the same derivatives at the point x_0. Then we can use the uniqueness theorem to conclude that they are the same function

$$f(x) = c_1u_1(x) + c_2u_2(x). \tag{3.28}$$

To show the existence of c_1, c_2 such that (3.27) holds we only need to prove that there is at least a point $x_0 \in I$ such that the system of equations (3.27) has a solution. As in the example, this is equivalent to finding at least a point $x_0 \in I$ such that the determinant

$$\begin{vmatrix} u_1(x_0) & u_2(x_0) \\ u_1'(x_0) & u_2'(x_0) \end{vmatrix} = u_1(x_0)u_2'(x_0) - u_1'(x_0)u_2(x_0) \neq 0. \tag{3.29}$$

In general, we define the determinant

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = u_1(x)u_2'(x) - u_1'(x)u_2(x), \tag{3.30}$$

and we call it the **Wronskian**.

Proving that there is at least a point $x_0 \in I$ such that the Wronskian $W(x_0) \neq 0$ is equivalent to proving that all solutions of the homogeneous equation (3.23) have the form (3.24). We will carry out this proof by proving that if we assume the contrary we arrive to a contradiction: Suppose that $W(x) = 0$ for all $x \in I$. Then this will imply that

$$\left(\frac{u_2}{u_1}\right)' = \frac{u_1u_2' - u_2u_1'}{u_1^2} = \frac{W}{u_1^2} = 0, \tag{3.31}$$

but this is in contradiction with one of the assumptions we made at the beginning, namely that u_2/u_1 is not a constant for $x \in I$. This means that the derivative of u_2/u_1 can not be vanishing, which is what would happen if the Wronskian is zero everywhere. Therefore $W(x_0) \neq 0$ for at least one point $x_0 \in I$.

Corollary: This theorem tells us that all solutions of the equation $L(y) = 0$ are of the form $c_1u_1(x) + c_2u_2(x)$, with c_1, c_2 being arbitrary constants. For this reason

$$y = c_1u_1(x) + c_2u_2(x), \tag{3.32}$$

is called the general solution of $L(y) = 0$. It follows also from the theorem that we can find the general solution by finding two particular solutions $u_1(x), u_2(x)$ such that $u_2/u_1 \neq$ constant. If $u_2/u_1 \neq$ constant we call u_1, u_2 **linearly independent solutions.**

Inhomogeneous equations:
Consider now the inhomogeneous equation

$$y'' + P_1(x)y' + P_2(x)y = R(x) = L(y), \tag{3.33}$$

with P_1, P_2, R being continuous functions on an open interval $I \in \mathbb{R}$. Suppose that y_1 and y_2 are two solutions of the inhomogeneous equation

$$L(y_1) = L(y_2) = R(x), \tag{3.34}$$

then by linearity of L

$$L(y_1 - y_2) = L(y_1) - L(y_2) = 0, \tag{3.35}$$

therefore $y_1 - y_2$ is a solution of the homogeneous equation $L(y) = 0$. However the previous theorem told us that the solutions of the homogeneous equation always are of the form $c_1u_1(x) + c_2u_2(x)$. Therefore $y_1 - y_2$ must be of the form

$$y_1 - y_2 = c_1u_1(x) + c_2u_2(x), \tag{3.36}$$

for some c_1, c_2, or equivalently

$$y_1 = c_1u_1(x) + c_2u_2(x) + y_2. \tag{3.37}$$

This proves that any pair of solutions of the inhomogeneous equation are related by (3.37). In other words, given a particular solution of $L(y) = R$, say y_1, all solutions of the differential equation are contained in the set

$$y = y_1 + c_1u_1(x) + c_2u_2(x), \tag{3.38}$$

where $L(u_1) = L(u_2) = 0$ and c_1, c_2 are arbitrary constants. For that reason (3.38) is called the **general solution** of the inhomogeneous equation (3.33).

Conclusion: Given an inhomogeneous equation

$$y'' + P_1(x)y' + P_2(x)y = R(x) = L(y), \tag{3.39}$$

with P_1, P_2, R being continuous functions on an open interval $I \in \mathbb{R}$, there are two problems we need to solve in order to find its general solution:

1. Find the general solution of the homogeneous equation $L(y) = 0$, i.e. $c_1u_1(x) + c_2u_2(x)$.
2. Find a particular solution of the inhomogeneous equation $L(y) = R$.

From last year's Calculus you already know some methods to solve simple homogeneous equations. Therefore we are going to start by learning a method to solve inhomogeneous equations: the method of variation of parameters.

3.1.2 The method of variation of parameters

Theorem: Let u_1, u_2 be two linearly independent solutions of a homogeneous equation $L(y) = 0$ on an interval I, with

$$L(y) = y'' + P_1(x)y' + P_2(x)y. \tag{3.40}$$

Let

$$W(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x), \tag{3.41}$$

be the Wronskian of u_1 and u_2. Then, the inhomogeneous equation $L(y) = R(x)$ has a solution y_1 given by

$$y_1(x) = v_1(x)u_1(x) + v_2(x)u_2(x), \tag{3.42}$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx, \tag{3.43}$$

Proof: Let us suppose that $y = v_1(x)u_1(x) + v_2(x)u_2(x)$ and try to determine $v_1(x)$ and $v_2(x)$. Then we just have to plug this solution into the equation

$$R(x) = y'' + P_1(x)y' + P_2(x)y, \tag{3.44}$$

and see what conditions must v_1, v_2 satisfy. Let us first compute

$$\begin{aligned} y' &= v_1'(x)u_1(x) + v_1(x)u_1'(x) + v_2'(x)u_2(x) + v_2(x)u_2'(x), && (3.45)\\ y'' &= v_1''(x)u_1(x) + v_1'(x)u_1'(x) + v_1'(x)u_1'(x) + v_1(x)u_1''(x) \\ &+ v_2''(x)u_2(x) + v_2'(x)u_2'(x) + v_2'(x)u_2'(x) + v_2(x)u_2''(x) \\ &= v_1''(x)u_1(x) + 2v_1'(x)u_1'(x) + v_1(x)u_1''(x) + v_2''(x)u_2(x) \\ &+ 2v_2'(x)u_2'(x) + v_2(x)u_2''(x), && (3.46) \end{aligned}$$

substituting these derivatives into (3.44) we obtain

$$\begin{aligned} R(x) &= v_1''(x)u_1(x) + 2v_1'(x)u_1'(x) + v_1(x)u_1''(x) + v_2''(x)u_2(x) \\ &+ 2v_2'(x)u_2'(x) + v_2(x)u_2''(x) + P_1(x)\left(v_1'(x)u_1(x) + v_1(x)u_1'(x)\right. \\ &+ \left.v_2'(x)u_2(x) + v_2(x)u_2'(x)\right) + P_2(x)(v_1(x)u_1(x) + v_2(x)u_2(x)), && (3.47) \end{aligned}$$

now we can use the fact that u_1, u_2 are solutions of the homogeneous equation, that is, they satisfy

$$\begin{aligned} 0 &= u_1''(x) + P_1(x)u_1'(x) + P_2(x)u_1(x), && (3.48)\\ 0 &= u_2''(x) + P_1(x)u_2'(x) + P_2(x)u_2(x). && (3.49) \end{aligned}$$

Using these equations we can prove that all terms in (3.47) which are proportional to $v_1(x)$ and $v_2(x)$ vanish, leaving us with the following condition

$$\begin{aligned} R(x) &= v_1''(x)u_1(x) + 2v_1'(x)u_1'(x) + v_2''(x)u_2(x) \\ &+ 2v_2'(x)u_2'(x) + P_1(x)\left(v_1'(x)u_1(x) + v_2'(x)u_2(x)\right) \\ &= (v_1'(x)u_1'(x) + v_2'(x)u_2'(x)) + (v_1'(x)u_1(x) + v_2'(x)u_2(x))' \\ &+ P_1(x)\left(v_1'(x)u_1(x) + v_2'(x)u_2(x)\right). && (3.50) \end{aligned}$$

One way of solving this equation is to impose further conditions on v_1, v_2 (remember, that we only need to obtain a particular solution of the inhomogeneous equation). For example, let us look for v_1, v_2 satisfying

$$v_1'(x)u_1(x) + v_2'(x)u_2(x) = 0 \quad \text{and} \quad v_1'(x)u_1'(x) + v_2'(x)u_2'(x) = R(x). \tag{3.51}$$

In this case we can solve these equations for v_1', v_2'. From the first equation we obtain

$$v_1'(x) = -v_2'(x)\frac{u_2(x)}{u_1(x)}, \tag{3.52}$$

substituting this in the second equation we get

$$\begin{aligned} -u_1'(x)v_2'(x)\frac{u_2(x)}{u_1(x)} + v_2'(x)u_2'(x) &= R(x) \\ \Rightarrow \quad v_2'(x) = \frac{u_1(x)R(x)}{u_2'(x)u_1(x) - u_1'(x)u_2(x)} &= \frac{u_1(x)R(x)}{W(x)}. \end{aligned} \tag{3.53}$$

Substituting this back into (3.52) we obtain

$$v_1'(x) = -\frac{u_2(x)R(x)}{W(x)}. \tag{3.54}$$

Integrating these equations we obtain the solutions (3.43). Let us see how this works with some examples:

Example 1: Solve the following 2nd order linear differential equation:

$$y'' + y = \tan x. \tag{3.55}$$

First we have to solve the homogeneous equation:

$$y'' + y = 0. \tag{3.56}$$

Last year you have seen that this kind of homogeneous equations with constant coefficients can always be solved by looking for solutions of the type $y = ce^{mx}$. Substituting this solution in (3.56) we obtain

$$m^2 + 1 = 0 \qquad \Rightarrow \qquad m = \pm i, \tag{3.57}$$

which means that the general solution of (3.56) can be written as

$$y = c_1 \cos x + c_2 \sin x, \tag{3.58}$$

therefore we identify

$$u_1(x) = \cos x \qquad \text{and} \qquad u_2(x) = \sin x. \tag{3.59}$$

Having this solution the next step is to solve the inhomogeneous equation (3.55). To do that we can use the method of variation of parameters which tells us that a particular solution of the inhomogeneous equation is given by:

$$y = v_1(x)u_1(x) + v_2(x)u_2(x) = v_1(x)\cos x + v_2(x)\sin x, \tag{3.60}$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx. \tag{3.61}$$

In our case

$$\begin{aligned} R(x) &= \tan x, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.62)\\ W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1. \end{aligned}$$

Therefore we have

$$v_1(x) = -\int \sin x \tan x dx = -\int \frac{\sin^2 x}{\cos x}dx, \tag{3.63}$$

we can solve this integral by changing variables as

$$t = \sin x \qquad \Rightarrow \qquad dt = \cos x dx, \tag{3.64}$$

which allows us to write

$$\begin{aligned} v_1(x) &= -\int \frac{t^2}{1-t^2}dt = -\int \frac{t^2-1+1}{1-t^2}dt \\ &= \int dt - \int \frac{1}{1-t^2}dt = t + \frac{1}{2}\int \left[\frac{1}{1-t} - \frac{1}{1+t}\right] dt \\ &= t + \frac{1}{2}\ln\left[\frac{t-1}{t+1}\right] + C, \end{aligned} \tag{3.65}$$

Recalling that $t = \sin x$ we obtain

$$v_1(x) = \sin x + \frac{1}{2}\ln\left[\frac{\sin x - 1}{\sin x + 1}\right] + C. \tag{3.66}$$

Similarly

$$v_2(x) = \int \cos x \tan x dx = \int \sin x dx = -\cos x + C' \tag{3.67}$$

Therefore, the general solution of (3.55) is

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x + \left[\sin x + \frac{1}{2}\ln\left[\frac{\sin x - 1}{\sin x + 1}\right]\right]\cos x - \cos x \sin x \\ &= c_1 \cos x + c_2 \sin x + \frac{\cos x}{2}\ln\left[\frac{\sin x - 1}{\sin x + 1}\right]. \end{aligned} \tag{3.68}$$

Notice that in the final solution we can forget about the constants C, C' because we are just interested in a particular solution of the inhomogeneous equation. Therefore, we can just choose them to be zero!

Example 2: Solve the equation

$$y'' + y = x^3. \tag{3.69}$$

You can solve this sort of equation without using the method of variation of parameters. In fact you have already solved equations of this type last year. Remember that whenever you have an equation of the form

$$y'' + ay' + by = R(x), \tag{3.70}$$

where a, b are constant coefficients and $R(x)$ is a polynomial of degree n, the solution to this equation is always of the form

$$y = a_1 + a_2 x + \ldots + a_n x^n, \tag{3.71}$$

with $a_1, a_2, \ldots, a_n$ constants.
Therefore a particular solution of (3.69) will be of the type

$$y = Ax^3 + Bx^2 + Cx + D, \tag{3.72}$$

with derivatives

$$\begin{aligned} y' &= 3Ax^2 + 2Bx + C, & (3.73)\\ y'' &= 6Ax + 2B, & (3.74) \end{aligned}$$

and if we substitute (3.72) and (3.74) in (3.69) we obtain

$$6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3, \tag{3.75}$$

from where it follows,

$$A = 1, B = 0, \quad 6A + C = 0 \Rightarrow C = -6, \quad B + 2D = 0 \Rightarrow D = 0. \tag{3.76}$$

So, our particular solution of the inhomogeneous equation is

$$y = x^3 - 6x, \tag{3.77}$$

and the general solution of the homogeneous equation is the same as in example 1. Therefore the general solution of (3.69) is

$$y = x^3 - 6x + c_1 \cos x + c_2 \sin x. \tag{3.78}$$

We could have solved this problem by using the method of variation of parameters. In that case the particular solution of the inhomogeneous equation would be

$$y = v_1(x) \cos x + v_2(x) \sin x, \tag{3.79}$$

with

$$\begin{aligned} v_1(x) &= -\int u_2(x) \frac{R(x)}{W(x)} dx = -\int x^3 \sin x, \\ v_2(x) &= \int u_1(x) \frac{R(x)}{W(x)} dx = \int x^3 \cos x. & (3.80) \end{aligned}$$

These integrals can be solved by parts (in fact we have to use integration by parts three times to solve each of the integrals!)

$$\begin{aligned} \int x^3 \sin x &= -x^3 \cos x + 3 \int x^2 \cos x dx \\ &= -x^3 \cos x + 3x^2 \sin x - 6 \int x \sin x dx \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \int \cos x dx \\ &= (6x - x^3) \cos x + (3x^2 - 6) \sin x, & (3.81) \end{aligned}$$

$$
\begin{aligned}
\int x^3\cos x &= x^3\sin x - 3\int x^2\sin x dx \\
&= x^3\sin x + 3x^2\cos x - 6\int x\cos x dx \\
&= x^3\sin x + 3x^2\cos x - 6x\sin x + 6\int \sin x dx \\
&= (x^3-6x)\sin x + (3x^2-6)\cos x. \qquad (3.82)
\end{aligned}
$$

Therefore the particular solution of the inhomogeneous equation would be

$$
\begin{aligned}
y &= v_1(x)\cos x + v_2(x)\sin x = -((6x-x^3)\cos x + (3x^2-6)\sin x)\cos x \\
&\quad +((x^3-6x)\sin x + (3x^2-6)\cos x)\sin x \\
&= (x^3-6x)(\cos^2 x + \sin^2 x) = x^3-6x. \qquad (3.83)
\end{aligned}
$$

Therefore, we obtain the same solution as with the other method but now we have to compute two quite lengthy integrals! The conclusion from this problem is that we must only use the method of variation of parameters when no other simpler method works. The method of variation of parameters is very powerful since it works for cases in which all other methods we know fail but we should not use it if we do not need to.

A Quadratic surfaces

In this appendix we will study several families of so-called quadratic surfaces, namely surfaces $z = f(x, y)$ which are defined by equations of the type

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Hx + Iy + Jz + K = 0, \tag{A.1}$$

with $A, B, C, D, E, F, H, I, J$ and K being fixed real constants and x, y, z being variables. These surfaces are said to be quadratic because all possible products of two of the variables x, y, z appear in (A.1).

In fact, by suitable translations and rotations of the x, y and z coordinate axes it is possible to simplify the equation (A.1) and hence classify all the possible surfaces into the following ten types:

1. Spheres
2. Ellipsoids
3. Hyperboloids of one sheet
4. Hyperboloids of two sheets
5. Cones
6. Elliptic paraboloids
7. Hyperbolic paraboloids
8. Parabolic cylinders
9. Elliptic cylinders
10. Hyperbolic cylinders

It is a requirement of this calculus course that you should be able to recognize, classify and sketch at least some of these surfaces (we will use some of them when doing triple integrals). The best way to do that is to look for identifying signs which tell you what kind of surface you are dealing with. Those signs are:

- The intercepts: the points at which the surface intersects the x, y and z axes.
- The traces: the intersections with the coordinate planes (xy-, yz- and xz-plane).
- The sections: the intersections with general planes.
- The centre: (some have it, some not).
- If they are bounded or not.
- If they are symmetric about any axes or planes.

A.1 Spheres

A sphere is a quadratic surface defined by the equation:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \tag{A.2}$$

the point (x_0, y_0, z_0) in the 3D space is the **centre** of the sphere. The points (x, y, z) in the sphere are all points whose distance to the centre is given by r. Therefore:

(a) The **intercepts** of the sphere with the x, y, z-axes are the points $x_0 \pm r, 0, 0)$, $(0, y_0 \pm r, 0)$ and $(0, 0, z_0 \pm r$.

(b) The **traces** of the sphere are circles or radius r.

(c) The **sections** of the sphere are circles of radius $r' < r$.

(d) The sphere is **bounded**.

(e) Spheres are symmetric about all coordinate planes.

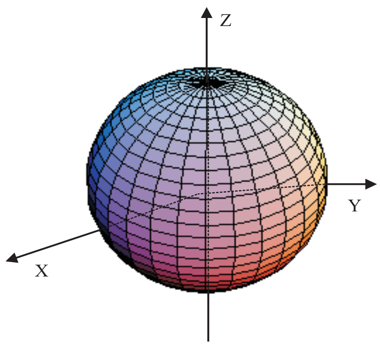
<u>Rule:</u> If we expand the square terms in equation (A.2) we obtain the following equation:

$$x^2 + y^2 + z^2 - 2x_0x - 2y_0y - 2z_0z + x_0^2 + y_0^2 + z_0^2 - r^2 = 0, \tag{A.3}$$

Comparing this equation with the general formula (A.1) we see it has the form

$$x^2 + y^2 + z^2 + Hx + Iy + Jz + K = 0, \tag{A.4}$$

with H, I, J and K being some real constants. Therefore, the rule to recognize a sphere is the following: any quadratic equation such that the coefficients of the x^2, y^2 and z^2 terms are equal and that no other quadratic terms exist corresponds to a sphere.



Sphere centered at the origin.

The sphere is a perfect example of a **surface of revolution**. A surface of revolution is a surface which can be generated by rotating a particular curve about a particular coordinate axis. For example, one way to generate the sphere of the picture above is to take the circle $x^2 + y^2 = 1$ and rotate it about the z-axis.

A.2 Ellipsoids

Ellipsoids are quadratic surfaces parameterized by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{A.5}$$

You can see that for $a = b = c = 1$ we recover the equation of a sphere centered at the origin. Therefore an ellipsoid is a "deformation" of the sphere such that the sphere gets either stretched or squeezed (depending on the values of a, b, c) in the x, y, z-directions.

(a) The **intercepts** of the ellipsoid with the x, y, z-axes are the points $(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$.

(b) The **traces** of the ellipsoid are ellipses which satisfy the equations:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{for } z = 0 \text{ (xy-plane)}, \tag{A.6}$$

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad \text{for } y = 0 \text{ (xz-plane)}, \tag{A.7}$$

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{for } x = 0 \text{ (yz-plane)}. \tag{A.8}$$

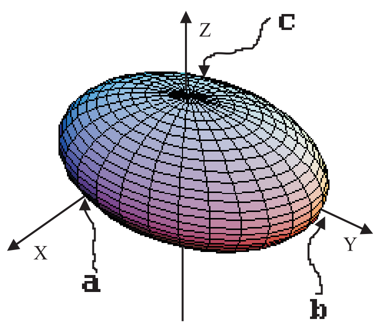
(c) The **sections** of the ellipsoid are also ellipses.

(d) The ellipsoid is **bounded** (all points (x, y, z) in the ellipsoid correspond to finite values of x, y and z.

(e) The **centre** of the ellipsoid in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(f) The ellipsoid is symmetric about all coordinate planes.

<u>**Rule:**</u> A quadratic equation such that the coefficients of the x^2, y^2 an z^2 terms are different from each other and all positive and such that no other quadratic terms exist corresponds to an ellipsoid.



Ellipsoid centered at the origin.

A.3 Hyperboloids of one sheet

A hyperboloid of one sheet is parameterized by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{A.9}$$

(a) The **intercepts** of the hyperboloid of one sheet with the x, y, z-axes are the points $(\pm a, 0, 0)$, $(0, \pm b, 0)$. Notice that there is no intersection with the z-axis. The reason is that if we set $x = y = 0$ in the equation (A.9) we obtain the condition $z^2 = -c^2$ which admits no real solution for real c.

(b) The **traces** of the hyperboloid of one sheet are ellipses in the xy-plane

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{for } z = 0 \ (xy\text{-plane}), \tag{A.10}$$

and hyperbolas in the xz- and yz-planes:

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.11}$$

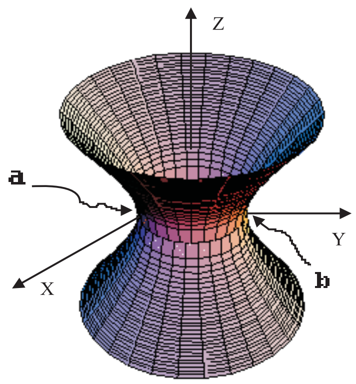
$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{for } x = 0 \ (yz\text{-plane}). \tag{A.12}$$

(c) The **sections** of the hyperboloid of one sheet are ellipses for planes parallel to the xy plane and hyperbolas for planes parallel to the yz- and xz-planes.

(d) The hyperboloid of one sheet is **not bounded**.

(e) The **centre** of the hyperboloid of one sheet in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(f) The hyperboloid of one sheet is symmetric about all coordinate planes.



Hyperboloid of one sheet centered at the origin.

A.4 Hyperboloid of two sheets

A hyperboloid of two sheets is a surface generated by the points satisfying the equation

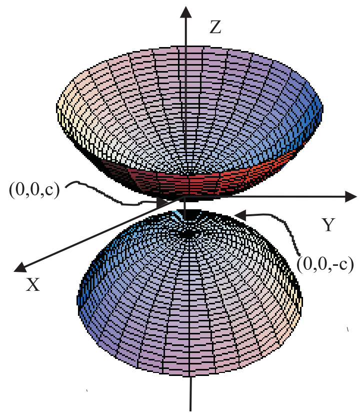
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1. \tag{A.13}$$

(a) The **intercepts** of the hyperboloid of two sheets with the x, y, z-axes are the points $(0, 0, \pm c)$. There are no intersections with the x, y-axes. The reason is that if we set $z = x = 0$ in the equation (A.9) we obtain the condition $y^2 = -b^2$ which can not be fulfilled for any real values of y and b. Analogously if we set $z = y = 0$ we obtain $x^2 = -a^2$ which also has no real solutions.

(b) In this case we have two sheets, in contrast to all examples we have seen so far. The reason is that the equation (A.13) implies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1. \tag{A.14}$$

This equation can only be solved if $z^2/c^2 - 1 \geq 0$, which implies that $|z| \geq c$.



Hyperboloid of two sheets centered at the origin.

(c) The **traces** of the hyperboloid of two sheets are hyperbolas in the xz- and yz-planes:

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.15}$$

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \quad \text{for } x = 0 \ (yz\text{-plane}). \tag{A.16}$$

(d) The **sections** of the hyperboloid of two sheets are hyperbolas for any planes parallel to the xz- or yz-planes and ellipses for planes parallel to the xy-plane with $|z| > c$.

(e) The hyperboloid of two sheets is also **not bounded**.

(f) The **centre** of the hyperboloid of two sheets in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(g) The hyperboloid of two sheets is again symmetric about all coordinate planes.

General rule: Any quadratic surface such that: the coefficients of x^2, y^2 and z^2 are different, one of the coefficients is negative and two of the coefficients are positive, and no other quadratic terms appear describes a hyperboloid. In addition, if the constant term has negative sign the hyperboloid has two sheets whereas if the constant term has positive sign the hyperboloid has only one sheet.

A.5 Cones

A cone is a quadratic surface whose points fulfil the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2 = 0. \tag{A.17}$$

Comparing (A.17) with the equations for the hyperboloids of one and two sheet we see that the cone is some kind of limiting case when instead of having a negative or a positive number on the l.h.s. of the quadratic equation we have exactly 0.

(a) The only **intercept** of the cone with the x, y, z-axes is the origin of coordinates $(0, 0, 0)$.

(b) The **traces** of the cone are lines in the xz- and yz-planes

$$z = \pm x/a \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.18}$$

$$z = \pm y/b \quad \text{for } x = 0 \ (yz\text{-plane}), \tag{A.19}$$

and the origin $(0, 0)$ in the xy-plane:

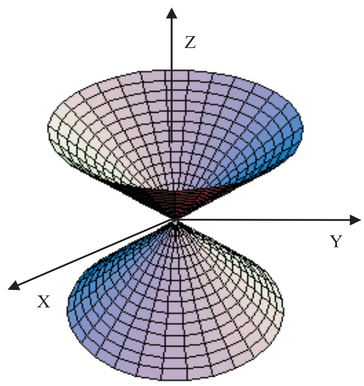
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \quad \text{for } z = 0 \ (xy\text{-plane}). \tag{A.20}$$

(c) The **sections** of the cone are lines for any planes parallel to the xz- pr yz-planes and ellipses (or circles if $a = b$) for planes parallel to the xy-plane.

(d) The cone is **not bounded**.

(e) The **centre** of the cone in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(f) The cone is symmetric about all coordinate planes.



Cone centered at the origin.

<u>Rule:</u> Any quadratic surface such that: the coefficients of x^2, y^2 and z^2 are different, one of the coefficients is negative and two of the coefficients are positive, no other quadratic terms appear and no constant term appears describes a cone.

A.6 Elliptic paraboloids

A quadratic surface is said to be an elliptic paraboloid is it satisfies the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z. \tag{A.21}$$

(a) The only **intercept** of the elliptic paraboloid with the x, y, z-axes is the origin of coordinates $(0, 0, 0)$.

(b) The **traces** of the paraboloid are parabolas in the xz- and yz-planes

$$z = \frac{x^2}{a^2} \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.22}$$

$$z = \frac{y^2}{b^2} \quad \text{for } x = 0 \ (yz\text{-plane}), \tag{A.23}$$

and the origin $(0, 0)$ in the xy-plane corresponding to the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \quad \text{for } z = 0 \ (xy\text{-plane}), \tag{A.24}$$

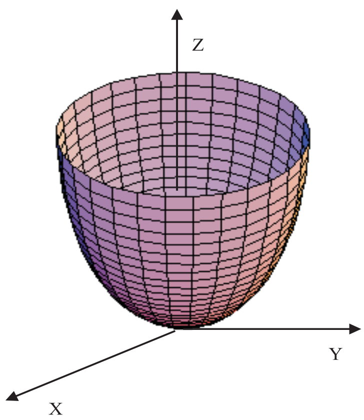
(c) The **sections** of the elliptic paraboloid with any planes parallel to the xz- or yz-planes are parabolas. The sections with planes parallel to the xy-plane are ellipses (or circles if $a = b$).

(d) The paraboloid is **not bounded** from above.

(e) The **centre** of the paraboloid in the picture is the origin of coordinates. It can be changed by shifting x, y, z by constant amounts.

(f) The elliptic paraboloid is symmetric about the xz- and yz-planes.

Rule: Any quadratic surface which contains: only linear terms in one of the variables (in our example z), quadratic terms in the other two variables with coefficients of the same sign and no constant term is an elliptic paraboloid.



Elliptic paraboloid centered at the origin.

A.7 Hyperbolic paraboloids

A hyperbolic paraboloid is defined by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z, \tag{A.25}$$

(a) The only **intercept** of the hyperbolic paraboloid with the x, y, z-axes is the origin of coordinates $(0, 0, 0)$.

(b) The **traces** of the paraboloid are parabolas in the xz- and yz-planes

$$z = \frac{x^2}{a^2} \quad \text{for } y = 0 \ (xz\text{-plane}), \tag{A.26}$$

$$z = -\frac{y^2}{b^2} \quad \text{for } x = 0 \ (yz\text{-plane}), \tag{A.27}$$

and two lines in the xy-plane corresponding to the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{for } z = 0 \ (xy\text{-plane}), \tag{A.28}$$

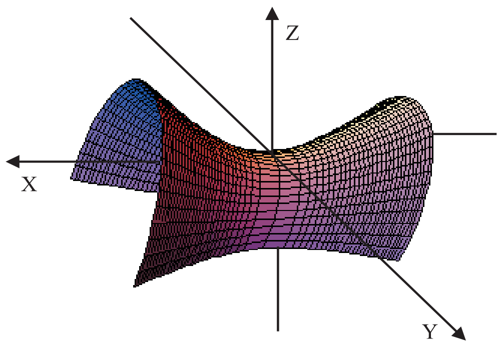
(c) The **sections** of the hyperbolic paraboloid with planes parallel to the xz- or yz-planes are parabolas. The sections with planes parallel to the xy-plane are hyperbolas.

(d) The paraboloid is **not bounded**.

(e) The **centre** of the paraboloid in the picture is the origin of coordinates which in this case is called a **saddle point**. It can be changed by shifting x, y, z by constant amounts.

(f) The hyperbolic paraboloid is symmetric about the xz- and yz-planes.

<u>**Rule:**</u> A hyperbolic paraboloid has the same features as the elliptic paraboloid with the only difference that the coefficients of the quadratic terms (x^2, y^2 in our example) have opposite signs.



Hyperbolic paraboloid centered at the origin.

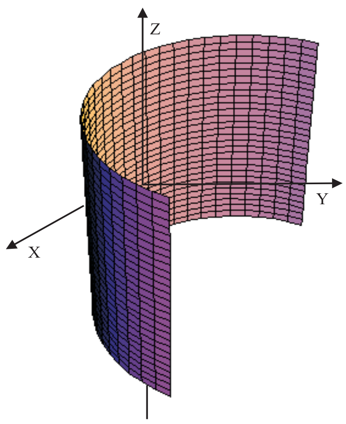
A.8 Parabolic cylinders

A parabolic cylinder is a quadratic surface whose points satisfy the equation

$$x^2 = 4cy. \tag{A.29}$$

This kind of surface is very simple, since the equation above does not depend on z. Since (A.29) describes a parabola in the xy-plane, the surface we are describing here is generated by the translation of the parabola (A.29) along the z-direction.

Rule: This kind of surface is easy to recognize, since its equation does not depend explicitly on one of the variables and is the equation of a parabola in the other two variables.



Parabolic cylinder centered at the origin.

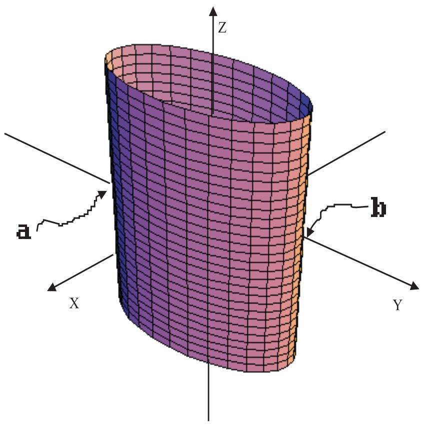
A.9 Elliptic cylinders

An elliptic cylinder is a quadratic surface described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{A.30}$$

As in the previous case, this surface does not depend explicitly on the coordinate z. Equation (A.30) describes an ellipse in the xy-plane (or a circle if $a = b$). Therefore an elliptic cylinder is the surface generated by the translation of the ellipse (A.30) along the z direction.

Rule: Again we have here a surface easy to recognize, since its equation does not depend explicitly on one of the variables and is the equation of an ellipse in the other two variables.



Elliptic cylinder centered at the origin.

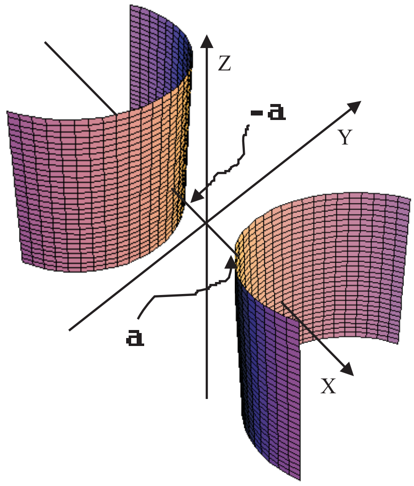
A.10 Hyperbolic cylinders

A hyperbolic cylinder is a quadratic surface parameterized by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{A.31}$$

The equation above describes a hyperbola in the xy-plane. Therefore this surface is simply generated by the translation of the hyperbola (A.31) along the z-direction.

Rule: The rule to recognize this kind of surface will be the same as in the previous case, with the difference that instead of an ellipse we have now a hyperbola.



Hyperbolic cylinder centered at the origin.

General rule: In general, cylinders are surfaces whose corresponding quadratic equation does not involve z explicitly. Therefore we must be told that they are in 3D to recognize a cylindrical surface.

References

[1] R. A. Adams, Calculus: A complete course (Addison Wesley).

[2] T. Apostol, Calculus (Wiley).

[3] S. N. Salas and E. Hille, Calculus: One and several variables (Wiley).

[4] H. Anton, Calculus (Wiley).