Dynamical Systems

Year 2

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Chapter 1

Introduction

1.1 Introduction

The course is split into two distinct parts, initially looking at first order or one dimensional systems and then moving on to two and possibly higher dimensional systems. In the introduction we give examples where we introduce first order systems in their own right as well as use the solution of a first order system to investigate a second order one.

The first natural question to ask is what is a *Dynamical System*.

We will start by defining a *Nonlinear System*. A Nonlinear System is a set of (one or more) nonlinear equations. Nonlinear equations are equations where the unknown quantity that we want to solve for appears in a nonlinear fashion. For example, if the quantity in question is a function $y(t)$, then terms such as y^2, $y\frac{dy}{dt}$ or $\sin y$ etc. would be nonlinear. *More precisely, a nonlinear equation is one where a linear combination of solutions is not a new solution.* If we denote $\dot{y} = \frac{dy}{dt}$, then the equation

$$\dot{y} = y(y-1), \tag{1.1}$$

is clearly nonlinear because of the y^2-term.

A nonlinear system may depend on one or several parameters (such as t in our example).

There are two main types of nonlinear systems: *differential equations* and *iterated maps*. You are already familiar with differential equations from previous years and equation (1.1) is an example of this type. In this module we will concentrate on nonlinear differential equations (although before that we will review a lot of results for linear equations!).

Differential equations describe nonlinear systems where the unknown function or functions depend on continuous variables (such as $t \in \mathbb{R}$ in (1.1)). In contrast, iterated maps arise in problems where the variables on which the solutions of the system depend take only discrete values. For example the equation:

$$x_{n+1} = rx_n(x_n - 1), \qquad \text{with} \qquad n = 0, 1, 2... \tag{1.2}$$

is an iterated map, since it depends on a discrete variable n. Here r is a constant. If we know the value of x_0 we can obtain x_1 by substituting x_0 on the r.h.s. of (1.2). Once we know x_1 we can find out x_2 by repeating the same "iterated" process.

A *dynamical system* is the same as a nonlinear system where the nonlinear equations represent the evolution of a solution with time or some variable that may be interpreted as time. For example, if we look at our equations (1.1) and (1.2) and interpret the variables t and n as time respectively, then we can think of these two equations as describing a dynamical system. Indeed, if we just think of the word dynamical, it suggests to us movement or change in time. The name dynamical originated in the context of physics, where nonlinear equations are very common.

Dynamical systems and nonlinear equations describe a great variety of phenomena, not only in physics, but also in economics, finance, chemistry and biology.

In this module we will mostly concentrate in learning the mathematical techniques that allow us to study and classify the solutions of dynamical systems. When possible, we will also emphasize the context in which the equations arise, namely, their interpretation.

Another way of defining a dynamical system is to say that it consists of a set of **States** and some **Rule** that takes us from one state to another in some prescribed way. In the examples (1.1) and (1.2) we can think of a "state" as the solution to the equations at a given time (e.g. a fixed value of t or n). Once we know such solution, we can use the equations to find out what the solution is for a later (or earlier) time.

Nonlinear equations are generally much harder to solve that linear ones. Most techniques for solving differential equations which you have seen in the last two years only really hold for linear equations. In fact, for many nonlinear equations it is not possible to find an explicit, analytic solution. They can only be solved numerically.

The additional complexity of nonlinear equations often leads to extremely interesting behaviour, such as the appearance of chaos and the relationship between nonlinear equations and fractals. For example, the fractal that you can find at http://en.wikipedia.org/wiki/Mandelbrot_set is called the Mandelbrot set and its picture can be generated by iterating the simple map $x_{n+1} = x_n^2 + c$, where c is a constant.

Similarly, if we iterate the map (1.2), we can find many interesting types of behaviour depending on the choice of the parameter r. For some values of r the system becomes **chaotic**, that is, it is extremely sensitive to small changes in initial conditions.

In this chapter I will introduce some of the main definitions/techniques that are used to study dynamical systems. I will do this by means of examples.

Example 1.1 - State of a system (*1 dimensional*)

Consider a population p governed by the differential equation

$$\frac{dp}{dt} = p\left(1 - \frac{p}{N}\right), \tag{1.3}$$

where t represents time and N is a constant[1]. Given the population at some time t_0 then the equation can be used to predict the population at some later (or even earlier) time t. This problem is one dimensional with a single state variable p, the rule that tell us how to move to another state is the differential equation Eq 1.3. Thus the time variable t is just a parameter that traces a path from one state to another. The set of all states is called the **phase space** or *state space*. A diagram of all

[1]This is in fact a very famous equation known as Logistic Equation. It provides the most simple realistic model for a population, as it accounts both for deaths and births. If we consider the simpler linear equation

$$\frac{dp}{dt} = p,$$

we will quickly see that it is solved by $p(t) = p_0 e^t$, where p_0 represents the initial population. Such as solution represents a population that grows exponentially in time. Any population with infinite resources and no predators would do so, but in most cases resources are limited and there are also predators, so it is sensible to modify the equation to account for this. This is achieved by introducing the term $-p^2$ on the r.h.s. This term represents a kind of competition between individuals of the same population which tends to decrease the population as time progresses.

the states indicating a direction corresponding to increasing t (*the independent variable*) is called a **phase diagram**.

The phase diagram for this problem is given in Fig 1.1 where we have only shown the region $p \geq 0$

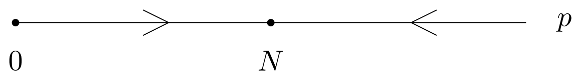


Figure 1.1:

since the system refers to a population which cannot realistically be negative. However a complete analysis of the system may well include all values of p. The direction of the arrows have been assigned to the diagram using the fact that if the derivative $\frac{dp}{dt} > 0$ then the function p is increasing with respect to the variable t ($\rightarrow$) and similarly if the derivative is negative it is decreasing ($\leftarrow$). Additionally if the derivative is zero then the function is said to be stationary. At such points we have constant (or fixed point) solutions. Indeed $p = 0$ and $p = N$ are constant solutions to the problem.

To sketch the diagram:

- First identify and insert the stationary (fixed)points as follows.

$$\frac{dp}{dt} = 0 \quad \Rightarrow \quad p(1 - \frac{p}{N}) = 0 \quad \Rightarrow \quad p = 0, \quad p = N$$

  Thus we draw a line with the two points $p = 0$ and $p = N$ indicated. (*usually with a full dot*)

- In the region $0 < p < N$ we can see that $\dfrac{dp}{dt} = p(1 - \dfrac{p}{N}) > 0$ thus p is increasing in this region. Hence the direction of the arrow in this region is from left to right $\rightarrow$.

- In the region $N < p$ we can see that $\dfrac{dp}{dt} = p(1 - \dfrac{p}{N}) < 0$ thus p is decreasing in this region. Hence the direction of the arrow in this region is reversed $\leftarrow$.

The diagram reveals, without any further attempt to solve Eq. (1.3), that for any initial population the long term trend is for the population to either increase or decrease to $p = N$. This is an example of a *limited growth* model and the point $p = N$ is called an *attractor* (we will see a proper definition of this later on). The interpretation of this is that if the initial population is small it will grow until a maximum is reached, which corresponds to an equilibrium between the amount of births and deaths. On the other hand, if the starting population is too large, they will compete with each other for resources, which will be scarce, so that for a while deaths will dominate births, until again an equilibrium is reached.

As you can see, one can actually say quite a lot about the solutions to a differential equation without necessarily solving it!

In this case, it is actually very easy to solve the equation, since we can completely separate variables $p(t)$ and t

$$\frac{dp}{dt} = p(1 - \frac{p}{N}) \quad \Leftrightarrow \quad \frac{N dp}{p(N-p)} = dt \quad \Leftrightarrow \quad \int \frac{N dp}{p(N-p)} = \int dt = t + c, \tag{1.4}$$

where c is an integration constant. Using partial fractions we have

$$\int \frac{N dp}{p(N-p)} = \int \left(\frac{1}{p} + \frac{1}{N-p}\right) dp = \log p - \log(N-p) = \log \frac{p}{N-p}. \tag{1.5}$$

Therefore

$$\log \frac{p}{N-p} = t + c \quad \Leftrightarrow \quad \frac{p}{N-p} = e^{t+c}, \tag{1.6}$$

which gives

$$p(t) = \frac{N}{1+e^{-t-c}}. \tag{1.7}$$

Hence, the population at time $t = 0$ is given by $p_0 = Ne^c/(1+e^c)$ in terms of the integration constant. Also, when $t \to \infty$, $p \to N$.

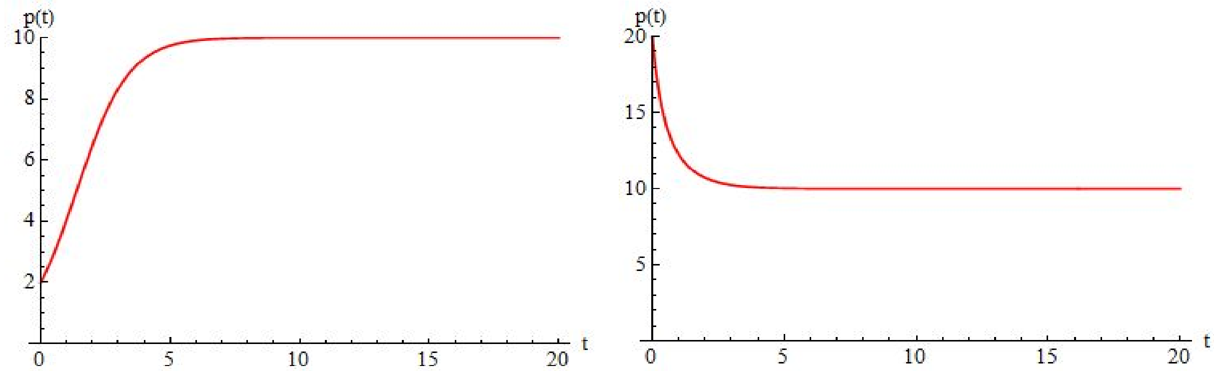


Figure 1.2: The figure on the l.h.s. represents the solution $p(t)$ for $p_0 = 2$. As expected $p(t)$ increases with time in this case until reaching the maximum value of N, which in this case has been fixed to $N = 10$. The figure on the r.h.s. corresponds to initial condition $p_0 = 20$. In this case the behavior is the opposite. The function $p(t)$ decreases with time until reaching the asymptotic value $N = 10$.

Definition
The equation

$$\frac{dy}{dt} = X(y)$$

is said to have a **fixed point** at $y = a$ if and only if $X(a) = 0$.
Clearly $y = a$ is also a solution of the equation. Fixed points are also referred to as **critical points**.

Example 1.2 - 2-dimensional problem - The Lodka-Volterra equations

Consider the following two dimensional problem that is used to model competition between two species (often a prey and a predator). Let x be the prey population and y the predator population and α, β, γ

and δ constants

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{1.8}$$

$$\frac{dy}{dt} = -\gamma y + \delta xy \tag{1.9}$$

The equations are of course an over simplification however they contain some of the basic ideas of modelling. In this simple model α may be seen as birth rate for preys and γ as a death rate for predators, respectively. The terms βxy and δxy represent the interaction between predators and pray. In general the population x will decrease due to such interaction, hence the negative sign in front of βxy. On the other hand the predators population y will generally increase as a result of interaction with prey, so the term δxy appears with a positive sign.

The question we now ask is what happens to the two populations as time evolves and what are the effects of α, β, γ and δ on this behaviour. This question is quite difficult to answer, however theoretically we could produce a phase diagram in two dimensions, that is a plot of y against x, by carrying out the following steps:

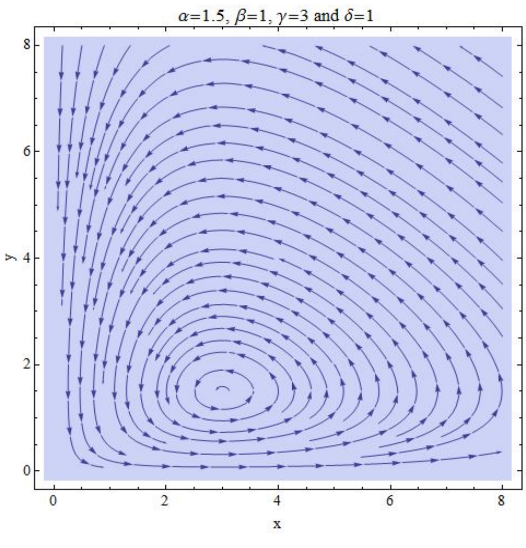
By dividing Eq(1.9) by Eq(1.8) and using $\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$ we obtain:

$$\frac{dy}{dx} = \frac{-\gamma y + \delta xy}{\alpha x - \beta xy} = \frac{y}{x}\left(\frac{-\gamma + \delta x}{\alpha - \beta y}\right) \tag{1.10}$$

This equation can be solved by separation of variables:

$$\int \frac{\alpha - \beta y}{y} dy = \int \frac{-\gamma + \delta x}{x} dx \quad \Rightarrow \quad \alpha \log y - \beta y \quad = \quad -\gamma \log x + \delta x + c, \tag{1.11}$$

Thus the phase diagram will be filled with the curves:



$$\alpha \log y - \beta y + \gamma \log x - \delta x = c. \tag{1.12}$$

For each value of c (and some fixed values of α, β, γ and δ) we get a curve. Unfortunately we can not solve for y as a function of x from this equation. However, this can be done on a computer leading to the phase diagram of Fig. 1.1.

A number of features of the solutions to the Lodka-Volterra equations can be read out from this phase diagram. A clear feature is that solutions are periodic. This is always the case when phase space trajectories are closed. It means that whatever the initial condition $(x(t_0), y(t_0))$ is, as t increases there will always come a time $t = t_0 + T$ when the solutions are the same as at t_0, that is $x(t_0 + T) = x(t_0)$ and $y(t_0 + T) = y(t_0)$. T is called the period.

It is easier to see this periodicity if we plot the solutions $y(t)$ and $x(t)$ to the equations, as functions of t (these are hard to compute, but can be obtained on a computer):

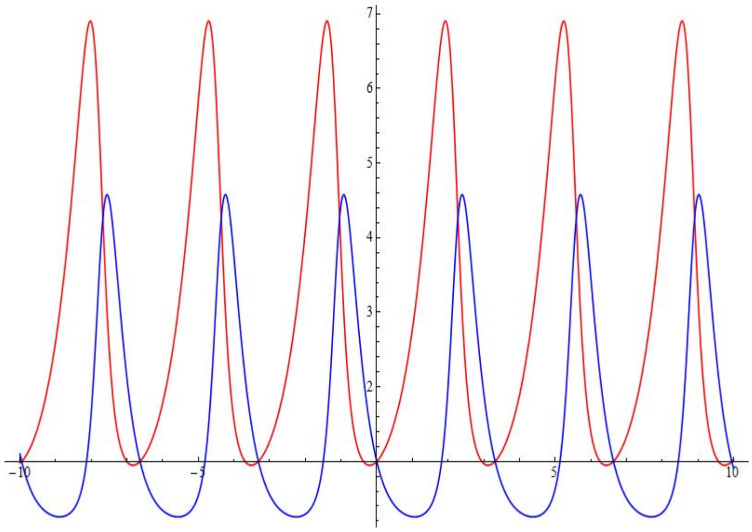


Figure 1.3: The red curve represents the function $x(t)$ and the blue curve represents the function $y(t)$ for some particular initial conditions $y(0) = x(0) = 1$. Clearly both functions are periodic in time. Also, as one would expect minima in prey numbers tend to coincide with maxima in predator numbers and viceversa.

Example 1.3 - Phase diagram for a simple 2-dimensional system

We now consider a system whose phase diagram can be obtained using the method described in the previous example.

Given:

$$\frac{dy}{dt} = -x \tag{1.13}$$

$$\frac{dx}{dt} = y \tag{1.14}$$

and applying the above method gives:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{x}{y}$$

Hence:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad \int y\, dy = \int -x\, dx \quad \Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C \quad \Rightarrow \quad y^2 + x^2 = 2C$$

From this we can conclude that phase space trajectories in this case are circles where the radius depends on the initial conditions (the constant C). For example, if we now consider the phase path through the point P given by $x = 1$, $y = 2$ then $C = 5/2$ and the phase path is given by $x^2 + y^2 = 5$ which is a circle centered at the origin with radius $\sqrt{5}$.

The trajectories in the phase diagram should also indicate the direction of increasing t. We can do this by looking at either of the original equations (1.13) or (1.14). Considering (1.14) for example we see that

$$\frac{dx}{dt} = y > 0 \qquad \text{for } y > 0,$$

which means that in the region of the phase diagram where $y > 0$ (the upper half-plane) $\frac{dx}{dt} > 0$, that is x is increasing as t increases. Thus the arrows in the phase diagram in the upper-half plane must point in the direction of increasing x, that is broadly towards the right $\rightarrow$.

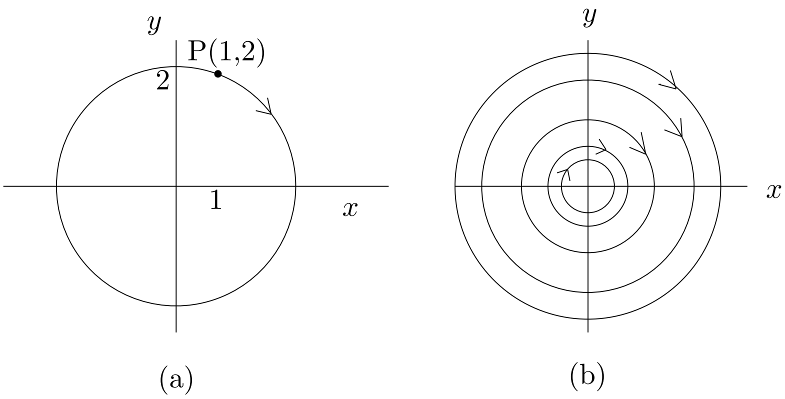


Figure 1.4: (a) Single phase path (trajectory) through P(1,2) (b) Phase diagram showing several such trajectories. The arrows indicate the direction of increasing t.

There are quite a lot of things that we can say about the solutions to equations (1.13)-(1.14) as functions of t, just by looking at the phase diagram. Since all trajectories are circles we can see that both $x(t)$ and $y(t)$ are periodic functions of t. The reason is that no matter where we start in the circle, as time progresses we will always eventually come back to the same point. We also see that whenever x reaches its maximum or minimum value we have $y = 0$ and, similarly, y is maximum or minimum when $x = 0$. This behaviour clearly reminds us of the periodic functions sine and cosine.

Indeed the solutions to equations (1.13)-(1.14) are

$$x(t) = A \sin t, \qquad\qquad y(t) = A \cos t, \tag{1.15}$$

where A is a constant that is fixed by the initial conditions. Clearly $x^2 + y^2 = A^2 =$ constant as we have already seen from our phase space analysis. In order for the point $P = (1, 2)$ to be part of the phase space trajectory we need $A^2 = 2C = 5$, which fixes $A = \sqrt{5}$. The value of t for which the solutions $x(t)$ and $y(t)$ pass by the point $(1, 2)$ can be obtained by solving:

$$\sqrt{5}\sin t = 1 \qquad \sqrt{5}\cos t = 2,$$

which gives

$$\tan t = \frac{1}{2},$$

that is $t = 0.463648$.

Example 1.4 - Second Order Differential System

We consider now a simple problem of a pendulum swinging about a smooth pivot. To simplify things the only mass in the problem is that of the bob at the end of the pendulum arm; the arm itself is considered to be rigid and have negligible mass compared with that of the bob. Gravity is considered to be the only force in the problem; friction due to the pivot and air resistance are ignored. (see Fig 1.5). The equation governing the motion of the pendulum can be written as:

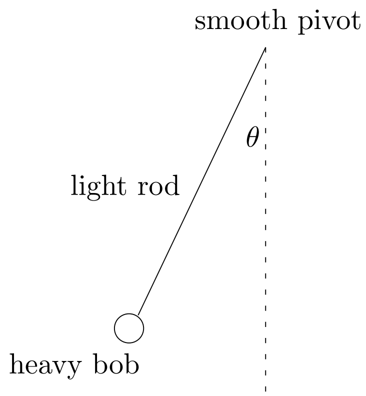


Figure 1.5: The pendulum swings under the influence of gravity, no other forces, such as friction, are considered. The angle theta is measured as positive in the clockwise direction

$$\frac{d^2\theta}{dt^2} = -\sin\theta \tag{1.16}$$

where t represents time.
The position of the pendulum is clearly given by the value of θ, however this alone is not sufficient to determine the state of the system. We would not say that a stationary pendulum with $\theta = \pi/4$, say, was in the same state as a moving pendulum with $\theta = \pi/4$. We thus need to introduce the speed at which the pendulum is swinging to completely describe the state of the system. This makes the problem a two dimensional system with two suitable state variablea being θ for position and $\dfrac{d\theta}{dt} = \dot{\theta}$ for speed.

To make this problem similar to the previous example we can use the following "trick": we set $x = \theta$ and $y = \dot{\theta}$. The single second order equation (1.16) can then be written as two first order equations as follows:

$$\frac{dx}{dt} = \frac{d\theta}{dt} = y \quad \text{and} \quad \frac{dy}{dt} = \frac{d^2\theta}{dt^2} = -\sin\theta = -\sin x \quad \Rightarrow$$

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -\sin x \tag{1.17}$$

Dividing the equations (1.17) by each other gives:

$$y\frac{dy}{dx} = -\sin x \quad \Rightarrow \quad \int y\,dy = -\int \sin x\,dx \quad \Rightarrow \quad y^2 - 2\cos x = C \tag{1.18}$$

The phase diagram for the system therefore consists of a collection of curves representing y against x according to (1.18), one for each value of C. They are plotted in Fig. 1.6

With reference to the phase diagram, if we trace the path of the point P as time evolves, P will

$\frac{dx}{dt} = y \quad \Rightarrow \quad x$ increasing when $y > 0$

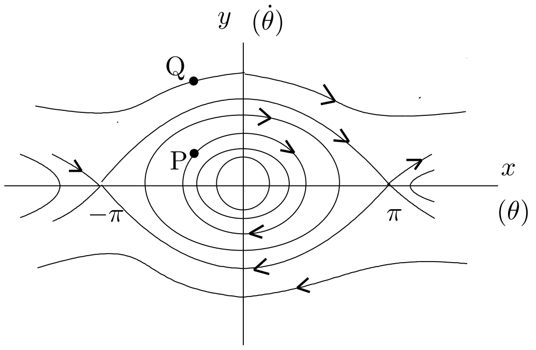


Figure 1.6: Phase diagram for $\ddot{\theta} = -\sin\theta$; arrows indicate direction of increasing t

in fact move round the path in a clockwise direction. At P, $x = \theta = -\pi/2$ and $y = \dot{\theta} = 1$, which represents the pendulum at the three o'clock position moving in a clockwise direction. Other points on this curve will represent other positions and speeds of the pendulum. At the point $(-\pi/2,\, 1)$ the integration constant in equation (1.18) gets fixed to $C = 1$ and the phase space trajectory is a closed trajectory corresponding to joining together the two functions:

$$y = \pm\sqrt{1 + 2\cos x}.$$

Each of these trajectories ends when $y = 0$, which in this case corresponds to $x = \pm 2\pi/3$. For larger values of x the function under the square root becomes negative and therefore there is no solution y

for such values of x. As in the previous example, when we had circular trajectories in phase space, we find once again that if our system starts at initial conditions corresponding to point P in the figure, it will eventually return to the same point (that is the original position and angular speed), representing a cyclic or swinging motion of the pendulum[2]

In contrast if the system starts at the point Q, θ does not return to its original value, indeed it continues to increase. This will represent a start up situation where the pendulum is given a sufficiently large value of $\dot{\theta}$ (*hit really hard to set it in motion*) to make it continually rotate about its pivot rather than just swing. In terms of the function (1.18) what this means is that the integration constant C is now sufficiently large so that the function y never becomes zero. An example would be to take again $x = \theta = -\pi/2$ and $y = \dot{\theta} = 3$. In this case $C = 3$ and we have again two solutions:

$$y = \pm\sqrt{3 + 2\cos x}.$$

Now the function under the square root is never zero and never negative. This means that in Fig. 1.6 the two solutions for y (with the plus and minus sign) do not meet anymore. Physically this means that if the pendulum starts with positive velocity $\dot{\theta}$ (clockwise direction) it will always have positive velocity (it will always move in the clockwise direction). If it starts with negative velocity it will always have negative velocity (anticlockwise direction).

Although the trajectories do not seem closed anymore, there is still periodicity in the sense that if we keep drawing our solution for larger or smaller values of x it will keep repeating itself (it oscillates periodically).

The phase diagram again gives us qualitative information regarding the system without actually solving the equation fully. Indeed obtaining θ in terms of t is not possible using elementary functions

Summary

Given:

$$\frac{dx}{dt} = X_1(x, y) \qquad \text{and} \qquad \frac{dy}{dt} = X_2(x, y)$$

the phase diagram for this system can be constructed by solving the first order equation:

$$\frac{dy}{dx} = \frac{X_2(x, y)}{X_1(x, y)}$$

Whether we are considering a one dimensional system given by a single first order differential equation or a second order system it will be essential that we can solve simple first order differential equations. The next section reviews two of the most common methods.

1.2 First Order Differential Equations

This section is split into three parts, namely the definition of what is meant by a solution to a differential equation, how to obtain a rough sketch of the solution and finally how to solve exactly two types of first order differential equation.

[2] You will see that the smaller the initial value of $y = \dot{\theta}$, the more the trajectories look like a perfect circle. This can be explained by noticing that for $x = \theta$ small, the original equation can be approximated by approximating the sine function as $\sin\theta \approx \theta$ (1st term in the Taylor expansion about $\theta = 0$). If we do this the system of equations (1.17) becomes exactly the same as in the previous example.

1.2.1 Definition of Solution

Definition

Let $X(x,y)$ be a real valued continuous function of the real variables x and y defined in some open [3] domain D of $\mathbb{R}^2$. A continuously differentiable function $y(x)$ with x in some interval I such that $\dfrac{dy(x)}{dx} = X(x, y(x))$ is said to be a solution of the equation $\dfrac{dy}{dx} = X(x,y)$.

Perhaps one of the most important facts to note is that even if $X(x,y)$ is a reasonable function a solution to the differential equation may, if it exists, only be defined on some interval of $\mathbb{R}$ rather than for all values of x. This definition always assumes that we are using the maximum possible interval when considering a solution.

As an example of this definition consider the problem solved in Eg 1.3, namely:

$$\frac{dy}{dx} = -\frac{x}{y} \qquad x = 1, y = 2$$

- Here $X(x,y) = -\dfrac{x}{y}$, hence a suitable domain is any set excluding $y = 0$. Indeed the largest possible domain is the set $D = \big\{(x,y) \in \mathbb{R}^2 : y \neq 0\big\}$.
- As we saw the solution was $y^2 + x^2 = 5$ hence $y = \pm\sqrt{5 - x^2}$ The definition of the solution requires that $y(x)$ is a function, which means that given x we must only have one value of y. Thus we cannot have a $\pm$ sign in the solution. Clearly $y = +\sqrt{5 - x^2}$, since with the negative sign y will not equal $+2$ at $x = 1$.
- Since $y = +\sqrt{5 - x^2}$, x is restricted to the range given by $0 \leq (5 - x^2)$ in order for the square root to be valid. Since $y \neq 0$, x is restricted further to $0 < (5 - x^2)$, that is to say, $-\sqrt{5} < x < \sqrt{5}$. The interval I therefore is given by $I = (-\sqrt{5}, \sqrt{5})$. [4]

Thus graphically the solution to this problem is the part of the circle in Fig 1.4(a) above the x-axis.

Example 1.5

Consider:

$$\frac{dy}{dx} = -\frac{y}{x} \qquad y = 1 \quad \text{and} \quad x = 1 \tag{1.19}$$

Solving gives:

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \quad \Rightarrow \quad \ln y = -\ln x + C$$
$$\Rightarrow \quad \ln(xy) = C \quad \Rightarrow \quad xy = e^C = A$$

Applying the conditions $x = 1$, $y = 1$ gives the solution as $y = \dfrac{1}{x}$. The graph of $y = \dfrac{1}{x}$ is in two halves, see Fig 1.7, however the only portion that gives a solution through $(1,1)$ is the one defined on the interval $I = (0, \infty)$.

[3] In $\mathbb{R}^2$ this can be thought of as an area that does not include its boundary

[4] The open brackets ( ) are used to indicate that the end points of the interval are omitted. To include the end point we use the square brackets [ ]. eg $(1,2)$ stands for the interval $1 < x < 2$ and $[1,2]$ for $1 \leq x \leq 2$

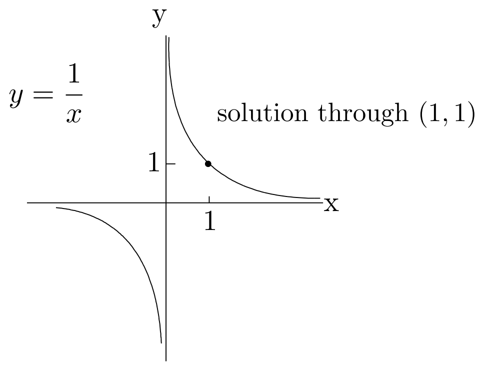


Figure 1.7: The solution to Eq 1.19 is the portion in the positive quadrant defined on the interval $I = (0, \infty)$

1.2.2 Solution by sketching

This is a method of obtaining a very rough sketch of the solutions to the equation $\dfrac{dy}{dx} = X(x, y)$. Recalling that $\dfrac{dy}{dx}$ is the gradient of the tangent to $y(x)$, we can use $X(x, y)$ to construct a line segment with gradient equal to that of a solution through any considered point. Consider the lattice of points $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1 \ldots n\}$ and at each point construct a small line segment with gradient equal to $X(x_i, y_i)$ and centred on the point. It therefore follows that each line segment is a small tangent to the solution through each point. Fig 1.8 shows the construction of such line segments

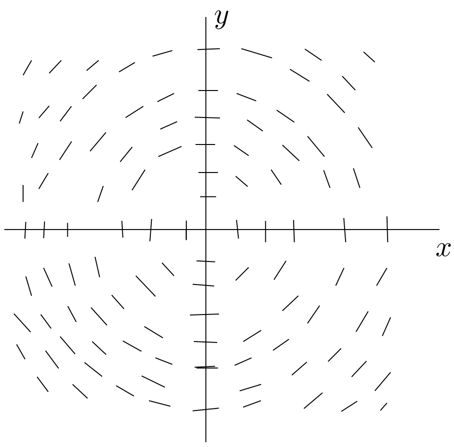


Figure 1.8: Velocity/vector field generated from $dy/dx = -x/y$. The trajectories can clearly be seen to be concentric circles about the origin as in Fig 1.4

for the system considered in Example 1.3 and illustrated in Fig 1.4. It is clear that from Fig 1.8 it is possible to sketch freehand the trajectories, ensuring at all times that the curves run in the same direction as nearby line segments. A good strategy for constructing the sketch is to start at the edge of the diagram between two line segments, drawing each curve with a slope somewhere between the slopes of each pair of segments.

Producing a diagram such as that on Fig 1.8 is a rather painstaking process and therefore is not very practical to do this by hand. That is why I have given you an Excel Macro that is going to do such drawings for you. You can get the file from the Moodle page.

1.2.3 Exact solution of first order differential equations

In this section we consider two elementary methods for solving first order differential equations. Namely the methods of **separation of variable** and **integrating factor**. As we shall see these can only be applied to a restricted set of equations but nevertheless prove to be extremely useful.

Separation of Variables We have already encountered this method above in Example 1.1 and Example 1.3. The general result is given by:

$$\frac{dy}{dx} = h(x)g(y) \quad \text{then} \quad \int \frac{dy}{g(y)} = \int h(x)\, dx \tag{1.20}$$

Clearly to obtain an exact solution to the problem it is now necessary to carry out the integration, this we know may not be a trivial task.

Example 1.6

Given

$$\frac{dy}{dx} = x \tan y \qquad\qquad D = \{(x, y) \in \mathbb{R}^2 :\ y \neq (2n+1)\pi/2,\ n = 0, \pm 1, \pm 2 \dots\}$$

find the solution such that $y = \pi/4$ at $x = 0$, carefully specifying its interval I of definition. Since the righthand side of the equation is the product of two functions, one in x and one in y we use the method of separation of variable to give:

$$\int \frac{dy}{\tan y} = \int x\, dx \quad \Rightarrow \quad \ln(\sin y) = \frac{x^2}{2} + C \quad \Rightarrow \quad y = \sin^{-1}(Ae^{x^2/2})$$

The condition $y = \pi/4$ at $x = 0$ gives the solution $y = \sin^{-1}\left(\dfrac{e^{x^2/2}}{\sqrt{2}}\right)$ where we have assumed the principal value of $\sin^{-1}$ in order to give $y = \pi/4$ at $x = 0$.

We now ask the question: is this expression valid for all values of x? The function $\sin^{-1}$ has domain equal to the interval $[-1, 1]$ which implies that:

$$-1 \leq \frac{e^{x^2}/2}{\sqrt{2}} \leq 1 \quad \Leftrightarrow \quad \frac{e^{x^2}/2}{\sqrt{2}} \leq 1 \quad \Leftrightarrow \quad e^{x^2/2} \leq \sqrt{2} \quad \Leftrightarrow \quad x^2 \leq 2\ln\sqrt{2} = \ln 2$$

Hence:

$$-\sqrt{\ln 2} \leq x \leq \sqrt{\ln 2}$$

Finally we note that since $y = \pm\pi/2$ is not in the domain of the problem we need to exclude the possibility of $x = \pm\sqrt{\ln 2}$.

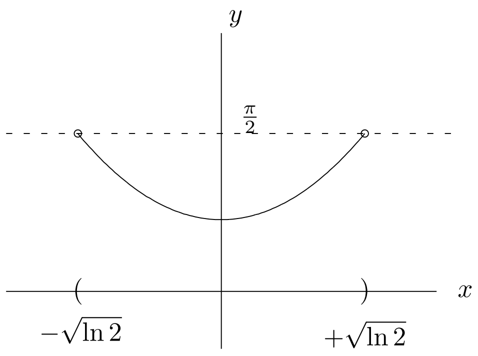


Figure 1.9:

Thus $I = (-\sqrt{\ln 2}, +\sqrt{\ln 2})$. See Fig 1.9

Integrating factor
We now consider the first order linear differential equation in the standard form: [5]

$$\frac{dy(x)}{dx} + P(x)y(x) = Q(x) \tag{1.21}$$

From Elementary Calculus we have that if we multiply this equation by $R(x) = \exp(\int P(x)\, dx)$ then we obtain:

$$\frac{d\, R(x)y(x)}{dx} = R(x)Q(x)$$

Hence

$$R(x)y(x) = \int R(x)Q(x)\, dx + C \quad \Rightarrow \quad y(x) = \frac{1}{R(x)}\left\{\int R(x)Q(x)\, dx + C\right\}$$

After carrying out the integration the arbitrary constant C is calculated using a given initial condition such as $y = y_0$ at $x = x_0$ that is to say $y(x_0) = y_0$.

An **alternative** method of automatically including the initial condition in the calculation rather than applying it later in order to determine C is to carry out a definite integral with the lower limit equal to x_0 and the upper limit set to the x. Doing this we obtain:

$$\frac{d\, R(x)y}{dx} = R(x)Q(x) \quad \Rightarrow \quad \int_{x_0}^{x} \frac{d\, R(x)y}{dx}\, dx = \int_{x_0}^{x} R(x)Q(x)\, dx$$

Hence

$$[R(x)y(x)]_{x_0}^{x} = \int_{x_0}^{x} R(x)Q(x)\, dx \quad \Rightarrow \quad R(x)y(x) - R(x_0)y_0 = \int_{x_0}^{x} R(x)Q(x)\, dx$$

Thus

$$y(x) = \frac{1}{R(x)}\left\{R(x_0)y_0 + \int_{x_0}^{x} R(x)Q(x)\, dx\right\}$$

[5]to emphasise the dependence on x we have denoted y as $y(x)$

The following example will be used to demonstrate both these methods;

Example 1.7

Given $y = \pi$ at $x = \pi$ solve the following differential equation:

$$\frac{dy}{dx} - \frac{y}{x} = x\cos x \qquad x \neq 0 \tag{1.22}$$

Calculating the integrating factor:

$$R(x) = \exp\left\{\int -\frac{1}{x}\,dx\right\} \quad = \quad \exp\{-\ln x\} \quad = \quad \frac{1}{x}$$

Thus multiplying Eq 1.22 by $R(x)$ gives:

$$\frac{d}{dx}\left(\frac{1}{x}\,y\right) \quad = \quad \frac{1}{x}\,x\cos x \quad = \quad \cos x$$

Thus

$$\frac{y}{x} = \int \cos x\;dx + C \quad \Rightarrow \quad y = x\sin x + Cx$$

The condition $y(\pi) = \pi$ implies that $C = 1$, thus the solution is given by $y = x(\sin x + 1)$. It appears that this solution is defined for all x, however the domain does not contain $x = 0$, thus for the solution to be defined on an interval that excludes $x = 0$ but includes $x = \pi$ we are restricted to $I = (0, \infty)$.

Using the **alternative** method:

$$\int_{\pi}^{x} \frac{d}{dx}\left(\frac{1}{x}\,y\right)\,dx = \int_{\pi}^{x} \cos x\;dx \quad \Rightarrow \quad \left[\frac{y(x)}{x}\right]_{\pi}^{x} = [\sin x]_{\pi}^{x}$$

Thus

$$\frac{y}{x} - \frac{\pi}{\pi} = \sin x - \sin\pi \quad \Rightarrow \quad y = x(\sin x + 1)$$

which is the same result as before.

1.2.4 Existence and uniqueness

We end this section with a short statement regarding the existence and uniqueness of our solution. Given an equation solutions may not always exist and if they do there may be more than one solution satisfying a given initial condition. A first order differential equation with the condition $y(x_0) = y_0$ is referred to as an **initial value problem**. The following theorem provides a set of sufficient conditions for the initial value problem to have a unique solution. The conditions of the theorem are not necessary, that is to say there may well be equations with unique solutions that do not satisfy all the conditions.

Theorem

Let $X(x, y)$ be defined on some open [6] domain $D \subseteq \mathbb{R}^2$ such that both $X(x, y)$ and $\dfrac{\partial X(x, y)}{\partial y}$ are continuous on D with the point $(x_0, y_0) \in D$. In such a case there exists a unique solution $y(x)$ of $\dfrac{dy}{dx} = X(x, y)$ defined on some interval I with $x_0 \in I$ such that $y(x_0) = y_0$.

We will not pursue this further but always assume that there does exist a unique solution to all our problems. As one would expect this theorem can be extended into higher dimensions.

[6] Roughly speaking we should think of an open set in $\mathbb{R}^2$ to be a region of $\mathbb{R}^2$ without its boundary eg $x^2 + y^2 < 1$ is the open set represented by the interior of a unit circle centred at the origin

Chapter 2

First Order Autonomous Dynamical Systems

2.1 Introduction

In chapter 1 we saw that a first order differential equation could either be used to describe a one dimensional problem such as the population model or be used to create a phase diagram for a two dimensional system as found in Example 1.3. In this chapter we look at the one dimensional systems in their own right and concentrate our attention to systems that do not depend explicitly on the independent variable. Such systems are referred to as autonomous.

Definition
The equation $\dfrac{dy}{dx} = X(y)$ is said to be autonomous when the righthand side does not depend explicitly on x.

Example 2.1

Consider the equation:

$$\frac{1}{k}\frac{dy}{dx} = a^2 - y^2 \qquad k > 0 \tag{2.1}$$

This equation can be used to model the free fall of an object through the air taking into account gravity and air resistance. In such a case y would stand for the speed of the object and x for the time of fall. Before trying to solve the equation we will construct its phase diagram by identifying regions where y is increasing, decreasing or neither. From such a diagram we are able to make certain qualitative statements about the system, such as given the value of y at some value of x what happens to y as x increases. The system is one dimensional thus the phase diagram is just a straight line.
The phase diagram Fig. 2.1 is constructed as follows:

- mark on the diagram the points where y is stationary. This occurs where $\frac{1}{k}\frac{dy}{dx} = a^2 - y^2 = 0$. Thus we mark $y = +a$ and $y = -a$
- In region (1) $y < -a$ thus $dy/dx < 0$ and hence y is decreasing with respect to x. This is indicated on the diagram by the first arrow $\leftarrow$.

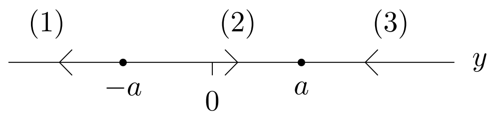


Figure 2.1: a is assumed to be positive. The three regions $y < -a$, $-a < y < a$ and $y > a$ are labelled (1), (2) and (3) respectively.

- In region (2) $-a < y < a$ thus $dy/dx > 0$ and hence y is increasing with respect to x. Again this is indicated using an arrow $\rightarrow$
- Similarly in region (3) y is decreasing which again is indicated by the arrow $\leftarrow$.

Fig 2.1 is the phase diagram for the system. We see that if the system starts with any value of y greater than $-a$ the system will tend to the fixed value $y = a$, either monotonically increasing or decreasing to this value as appropriate. With an initial value of y less than $-a$ the system moves away from $-a$ towards $-\infty$. The precise interpretation of the diagram will always depend on the physical situation being modelled. For the falling object problem the equation is actually only valid for $y \geq 0$ thus the fact that all starting values greater or equal to zero tend to $+a$ is interpreted physically as the terminal velocity of a falling object. ie all falling objects reach a maximum speed known as the terminal velocity.

The phase diagram can also be helpful in sorting out the different types of solutions obtained by starting in each of the three regions. In this problem we can actually obtain the exact solution to the differential equation, though in most realistic problems this is not possible and we need to resort to a numerical method. In such a case the phase diagram is then useful in providing us with prior knowledge of the solution.

Exact solution

Using the general starting condition $y(x_0) = y_0$ and the method of separation of variables, integrating Eq. (2.1) gives:

$$\int_{y_0}^{y} \frac{dy}{a^2 - y^2} = \int_{x_0}^{x} k\, dx \quad \Rightarrow \quad \frac{1}{2a} \int_{y_0}^{y} \left(\frac{1}{a-y} + \frac{1}{a+y} \right) dy = \int_{x_0}^{x} k\, dx$$

Which on integration gives:

$$\left[\frac{1}{2a} \ln \left(\frac{a+y}{a-y} \right) \right]_{y_0}^{y} = [kx]^{x}{}_{x_0}$$

Hence

$$\ln \left(\frac{a+y}{a-y} \right) - \ln \left(\frac{a+y_0}{a-y_0} \right) = 2ka(x - x_0)$$

$$\Rightarrow \quad \ln \left\{ \left(\frac{a+y}{a-y} \right) \left(\frac{a-y_0}{a+y_0} \right) \right\} = 2ka(x - x_0)$$

$$\Rightarrow \quad \frac{a+y}{a-y} = \left(\frac{a+y_0}{a-y_0} \right) e^{2ka(x-x_0)}$$

After some manipulation this gives:

$$y = a\frac{\{(a + y_0)e^{2ka(x-x_0)} - (a - y_0)\}}{\{(a + y_0)e^{2ka(x-x_0)} + (a - y_0)\}} \tag{2.2}$$

Our task now is to plot the solution obtained in Eq 2.2 for values of y_0 in each of the three regions identified in Fig 2.1.

First however note the following:

- The solution depends on $(x - x_0)$, ie the difference between x and its starting value. If in an example x represented time, the solution is seen to only depend on the amount of time elapsed from the beginning and not the actual time on your clock or date on your calendar. This property is a consequence of the problem being autonomous, indeed it can be proved that all solutions of an autonomous system can be expressed in terms of $(x - x_0)$. Graphically this means that all the solutions will *essentially* be the same; in each of the three regions any one solution is a translation of the other.

- With k and a assumed to be greater than zero the formula in Eq. (2.2) implies that y tends to $+a$ as x tends to infinity and to $-a$ as x tends to minus infinity for all values of y_0, with of course the exception of $y_0 = \pm a$, which are constant solutions. However as seen below since a solution of a differential equation can only be defined on an interval it may not be meaningful to carry out these limits for some starting values y_0.

Graph in region (2) $-a < y_0 < +a$
With y_0 in this range the denominator of the solution in Eq. (2.2) can never be zero as it consists of the sum of two positive terms for all values of x. Thus the interval of definition in this range is given by $I = (-\infty, +\infty)$. The graph is given in Fig 2.2; we note that the solution has the expected behaviour as $x \to \pm\infty$.

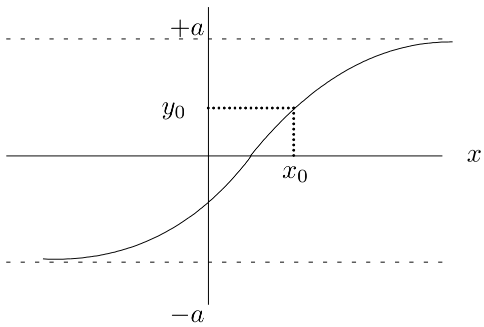


Figure 2.2: Solution of Eq. (2.2) for $-a < y_0 < a$; $I = (-\infty, +\infty)$

Graph in region (3) $a < y_0$
In this region $(a - y_0)$ is negative and $(a + y_0)$ is positive, hence the denominator of the solution in Eq 2.2 will vanish at some value of x. Denoting this value of x as x_∞ it is given by:

$$(a + y_0)e^{2ka(x_\infty - x_0)} + (a - y_0) = 0$$

Solving for x_∞ this gives:

$$x_\infty = x_0 + \frac{1}{2ak} \ln \left(\frac{y_0 - a}{y_0 + a} \right) \tag{2.3}$$

Also in this range we observe that $y_0 - a < y_0 + a$ thus $\frac{y_0 - a}{y_0 + a} < 1$ which means that the ln term in Eq. (2.3) is negative and hence we can deduce that x_∞ is to the left of x_0. The interval of definition can not include x_∞ but must include x_0, therefore $I = (x_\infty, \infty)$(see Fig. 2.3). With this interval of definition it is not meaningful to consider the limit of Eq. (2.2) as $x \to -\infty$. Indeed as x decreases from x_0 towards x_∞ the solution y tends to $+\infty$ and not $-a$. The limit of Eq. (2.2) as $x \to +\infty$ is meaningful. As expected, and as can be seen in Fig. (2.3), y tends to $+a$ as x tends to ∞.

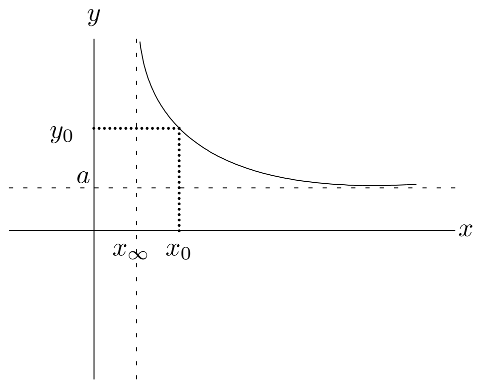


Figure 2.3: *Solution of Eq 2.2 for $a < y_0$; $I = (x_\infty, \infty)$*

Graph in region (1) $y_0 < -a$
In this region $(y_0 + a)$ is negative and $(a - y_0)$ is positive, hence the denominator of the solution in Eq 2.2 will again vanish for some value of x. As before, denoting it as x_∞, it is given by Eq 2.3. However this time $\frac{y_0 - a}{y_0 + a} > 1$ and hence the ln term in Eq 2.3 will be positive and hence x_∞ will be to the right of x_0. In this case the interval of definition is given by $I = (-\infty, x_\infty)$ See Fig 2.4.

We now make the following observations:

- The starting value y_0 clearly influences the solution greatly, changing its shape and interval of definition.
- The solution depends on $(x - x_0)$ which means that the solutions do not change shape as we vary x_0 but simply translate to the right or left. This property is common to all autonomous systems. Physically, treating x as time, it is saying that the system performs in an identical manner no matter when you start.
- The points $y = -a$ and $y = +a$ are special to this problem. Not only are they the values of y at which $\frac{dy}{dx} = 0$ but they are also constant (fixed) solutions of the equation. Clearly the original equation Eq 2.1 is satisfied by substituting $y = \pm a$ into both sides.

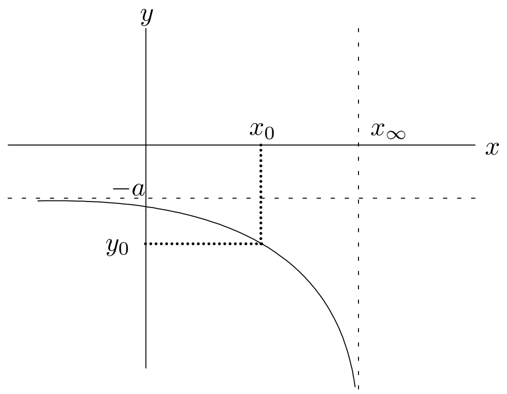


Figure 2.4: *Solution of Eq 2.2 for $y_0 < -a$; $I = (-\infty, x_\infty)$*

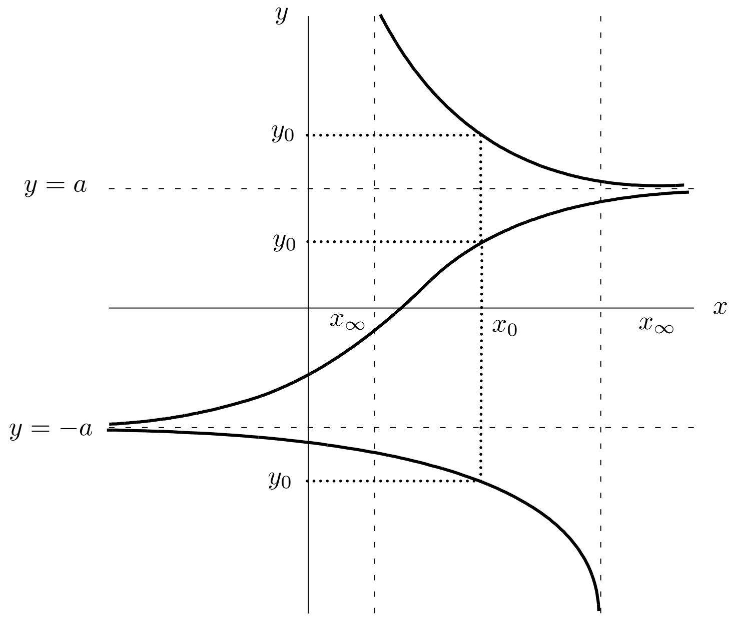


Figure 2.5: *Complete Solution of Eq. (2.2) with y_0 shown in each of the three regions.*

2.2 Classification of Fixed Points

In the introduction, Sec. 2.1, we considered an example which demonstrated two different types of fixed point. The one at $y = a$ attracted solutions, in this sense it is considered as stable. The other fixed point at $y = -a$ repelled solutions and will be thought of as unstable. To consolidate these ideas we make the following definitions.

Definition - stable

A fixed point P [1] is **stable** if **all** trajectories[2] that start close to P stay close to P as x increases.

Additionally a fixed point is **asymptotically stable** if all trajectories close to P tend to P as $x \to \infty$.

Definition - unstable
A fixed point is unstable if it is not stable.

The above definitions are not rigorous but they are applicable to fixed points of any dimensional autonomous system.

Definition - attractor

For a one dimensional system an asymptotically stable fixed point is called an **attractor**.

Definition - repellor
For a one dimensional system a fixed point P such that all trajectories close to P move away from P, as x increases, is called a **repellor**.

Definition - shunt
For a one dimensional system a fixed point P such that in every neighbourhood of P some trajectories are attracted to P and some are repelled by P is called a **shunt**

In the final example of Sec. 2.1 the fixed point at $y = +a$ is an attractor and is therefore stable; the fixed point at $y = -a$ is a repellor and therefore unstable.

Example 2.2

Obtain and classify the fixed points of $\dfrac{dy}{dx} = y^2(y^2 - 1)$. Hence draw the phase diagram.

- The fixed points are located by solving $y^2(y^2 - 1) = 0$ which gives $y = 0$, $y = +1$ and $y = -1$. Enter these onto the phase diagram in Fig 2.6. This splits the phase diagram into four distinct regions.
- In region (1) $dy/dx = y^2(y^2 - 1) > 0$, thus y is increasing $\rightarrow$.
- In regions (2) and (3) $dy/dx = y^2(y^2 - 1) < 0$, thus y is decreasing in both these regions $\leftarrow$.
- In region (4) $dy/dx = y^2(y^2 - 1) > 0$, thus y is again increasing as indicated by the direction of the arrow $\rightarrow$.

Thus we see from the direction of the arrows that $y = -1$ is an **attractor** and therefore **stable**; $y = +1$ is a **repellor** and therefore **unstable**. The fixed point at $y = 0$ is a mixture of attractor and repellor in all its neighbourhoods, therefore it is a **shunt**. A **shunt is not stable** since not all trajectories close to it stay close. Indeed as we see, if y is to the left of 0 it will move way from 0 towards $-\infty$.

[1] P is a point in the phase plane; in 1-dimension P is a point on the straight line, in two dimensions it is a point in a plane.

[2] a trajectory is a path in the phase plane; in 1-dimension a trajectory is a part of the real line, in 2-dimensions it is a curve, both depend on the independent variable in the problem.

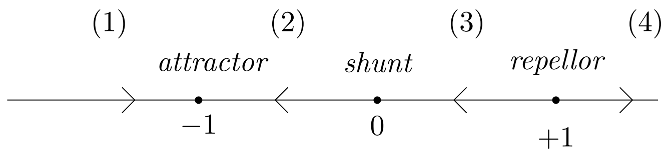


Figure 2.6: Phase diagram for $dy/dx = y^2(y^2-1)$

Additionally we remark that $y = 0$, $y = 1$ and $y = -1$ are solutions of the equation.

2.3 Linearisation of 1-dimensional systems

Classifying fixed points and obtaining solutions to a given problem can in general be quite difficult. We look now at a method that enables us to consider the system in the neighbourhood of its fixed points. By considering a simplified version of the problem in each of these neighbourhoods we can classify the fixed points and obtain approximations to the solutions.

2.3.1 The Linear Equation

Consider the equation:

$$\frac{dy}{dx} = X(y) \tag{2.4}$$

and let $y = a$ be a fixed point; ie $X(a) = 0$. Let us introduce a new variable $z = y - a$, that is $y = z + a$. We can now rewrite the original differential equation in terms of this new variable z as:

$$\frac{dz}{dx} = X(z+a). \tag{2.5}$$

Since we are interested in studying the behaviour of the system near the fixed point (y very close to a), this means that the values of z we are interested in are very small (since $z = y - a$). Therefore, it makes sense to expand $X(z + a)$ using Taylor's expansion about $z = 0$.

Thus:

$$\frac{dz}{dx} = X(z+a) = X(a) + zX'(a) + \frac{z^2}{2}X''(a) + \ldots$$

Since we wish to look at the problem close to $y = a$ we will ignore terms in powers of z^2 and above. Also since a is the fixed point $X(a) = 0$. Thus the linearisation of Eq 2.5 about $y = a$ becomes:

$$\frac{dz}{dx} = zX'(a) \tag{2.6}$$

This differential equation has a solution which is defined for all x, that is to say the interval of definition for the linearised case is $(-\infty, +\infty)$. This may be different from the original problem where solutions may have only been defined on some interval I of the real line. The general solution to (2.6) is simply:

$$z = Ae^{X'(a)x}, \tag{2.7}$$

where A is a constant that depends on the initial conditions of the problem.

Once we have the solution for z we can find the solution for y by simply using the definition $y = z + a$, so

$$y = a + Ae^{X'(a)x}. \tag{2.8}$$

Example 2.3

As a first example consider the Eq. (2.1) at the beginning of this chapter with $a = 2$ and $k = 1$, namely, $\dfrac{dy}{dx} = 4 - y^2$. It has two fixed points at $y = \pm 2$. Consider now the linearisation of this equation about each of these points in turn. With reference to Eq. (2.5), in this example $X(y) = 4 - y^2 \quad \Rightarrow \quad X'(y) = -2y$. Thus at each of the two fixed points we have:

- About $y = 2$.

  About this point $y = 2 + z$ and the linearisation is given as:

  $$z' = X(2)z \quad \Rightarrow \quad z' = -4z \quad \Rightarrow \quad z = Ae^{-4x} \quad \Rightarrow \quad y = 2 + Ae^{-4x}$$

  Introducing the initial condition $y = y_0$ at $x = x_0$ and calculating A gives:

  $$y = 2 + (y_0 - 2)e^{-4(x - x_0)}$$

  Consider now the point P in Fig.(2.7). The curves that go by P represent the exact and linearised solutions to the equation for $(y_0 - 2) > 0$. In this region the linearised solution approaches 2 from above as $x \to \infty$ and tend to $+\infty$ as $x \to -\infty$. Note that as stated above this is defined for all x. Similar remarks apply if the linear solution is started in either of the other two regions.

- At $y = -2$

  About this point $y = -2 + z$ and the linearisation is given as:

  $$z' = X(-2)z \quad \Rightarrow \quad z' = 4z \quad \Rightarrow \quad z = Ae^{4x} \quad \Rightarrow \quad y = -2 + Ae^{4x}$$

  Introducing the initial condition $y = y_0$ at $x = x_0$ and calculating A gives:

  $$y = -2 + (y_0 + 2)e^{4(x - x_0)}$$

  We now consider this solution for the case $y_0 < -a$, the point Q in Fig.(2.7). As $(y_0 + 2) < 0$ the solution approaches -2 from below as $x \to -\infty$ and tend to $-\infty$ as $x \to +\infty$.

Note that both these curves are shown plotted in Fig.(2.7). The important thing to note is that a linear solution is only close to an exact solutions when it is close to its associated fixed point. For example the linear solution about $y = 2$ through P is seen only to be close to the exact solution near $y = 2$.

Example 2.4

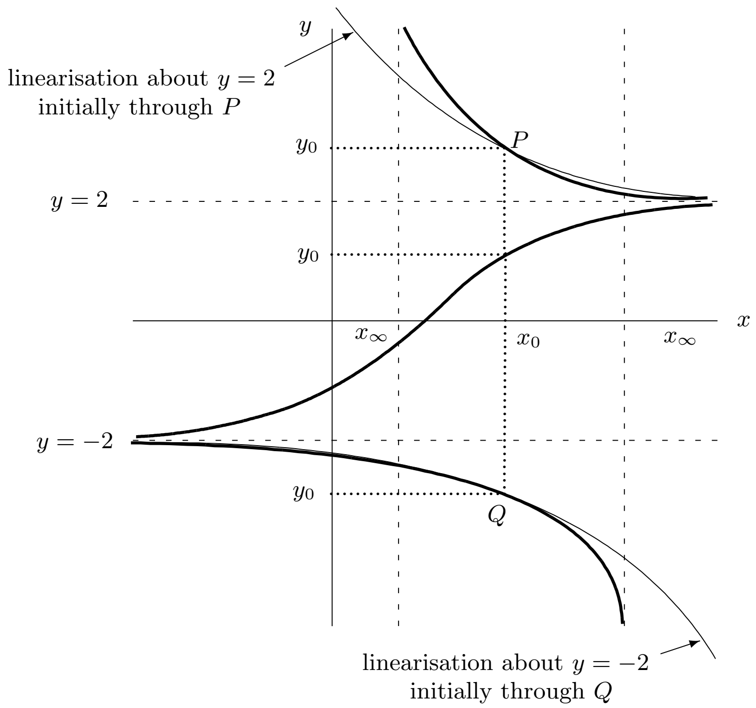


Figure 2.7: *Solutions of $dy/dx = 4 - y^2$; 3 starting values for $y(x_0) = y_0$ in the three regions $y_0 < -2$, $-2 < y_0 < 2$ and $2 < y_0$; two linearisations, one about $y = 2$ initially passing through P and one about $y = -2$ initially passing through Q; Exact solution shown in thick line, linear approximations in thin line, asymptotes in dashed line.*

As a second problem consider the population model from Chapter 1, namely $\dfrac{dp}{dt} = p\left(1 - \dfrac{p}{10}\right)$ with $N = 10$. If all values of p are considered this has an attractor at $p = 10$ and a repellor at $p = 0$. The phase diagram is given by Fig 2.8 The phase diagram for this problem is the same as Fig 2.1. If the differential equation is solved in the three regions $p < 0$, $0 < p < 10$ and $10 < p$, three different solutions are obtained. Again not all the solutions are defined for all values of t.

The solution satisfying the general initial condition $p(t_0) = p_0$ is given by:

$$p = \frac{10p_0e^{t-t_0}}{(10 - p_0) + p_0e^{t-t_0}}$$

Again we see that the solution depends on $t - t_0$, a property of the equation being autonomous. Values of t_∞ are found by equating the denominator equal to zero for p_0 in each of the three regions. The graph of all the solutions is given in Fig. 2.9

Let us now consider the linearisation of this problem about its two fixed points. In this example $X(p) = p(1 - p/10)$ thus $X'(p) = 1 - p/5$ and hence $X'(0) = 1$ and $X'(10) = -1$. From the sign of $X'(p)$ we can say immediately that $p = 0$ is a repellor and $p = 10$ is an attractor. We now look at the solutions of the linearised problem:

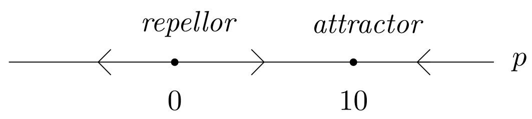


Figure 2.8: Phase diagram for $dp/dt = p(1 - p/10)$

- About the point $p = 0$:

$$z' = X'(0)z = z \quad \Rightarrow \quad z = Ae^t$$

Given that $p = p_0$ at $t = t_0$ and remembering that in this case $p = z$, we get the linearised solution:

$$p = p_0 e^{(t-t_0)}$$

This is plotted in Fig 2.9 with the initial point being at Q (*R or P could equally well been chosen as the initial point to illustrate this solution*). Note that the exact solution and this linear approximation get closer as p approaches zero.

- About the point $p = 10$:

$$z' = X'(10)z = -z \quad \Rightarrow \quad z = Ae^{-t}$$

Given that $p = p_0$ at $t = t_0$ and remembering that in this case $p = z + 10$, we get the linearised solution:

$$p = 10 + (p_0 - 10)e^{-(t-t_0)}$$

This is plotted in Fig 2.9 where, in this case, the initial point is taken to be at P. Note that the exact solution and this linear approximation get closer as p approaches 1.

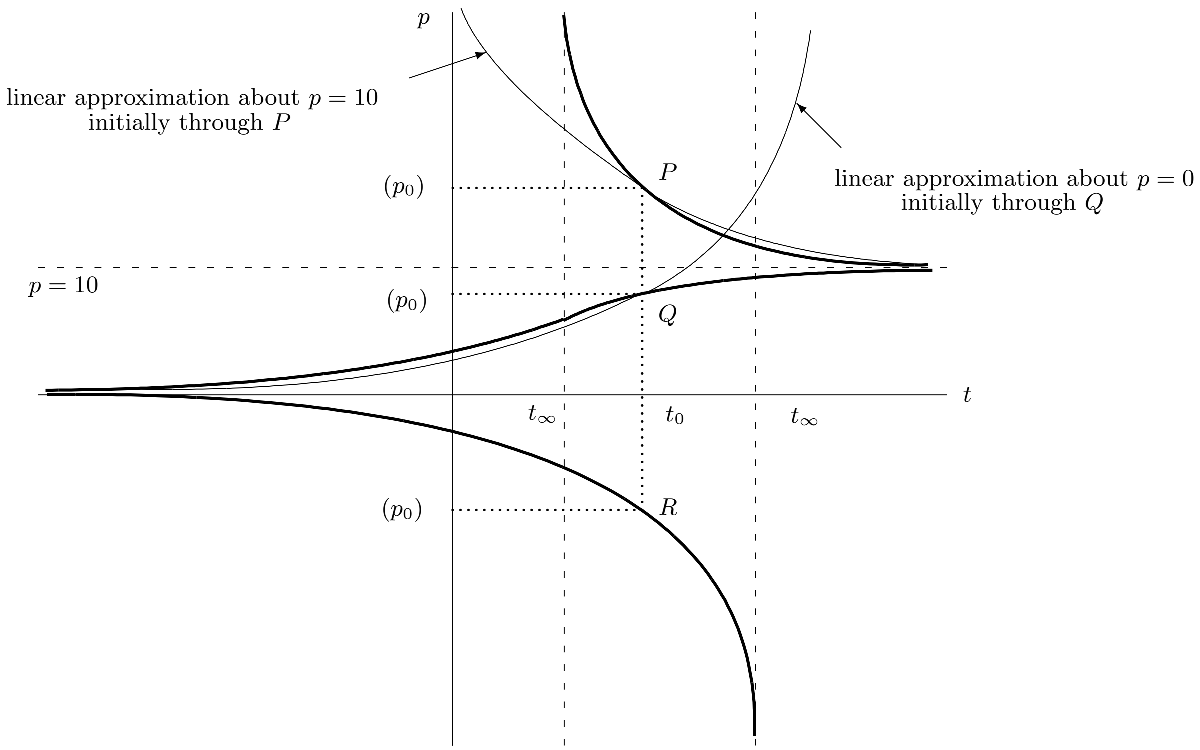


Figure 2.9: *Solutions of $dp/dt = p(1 - p/10)$; 3 starting values for $p(t_0) = p_0$ in the three regions $p < 0$, $0 < p < 10$ and $10 < p$; two linearisations, one about $p = 10$ initially passing through P and one about $p = 0$ initially passing through Q; Exact solution shown in thick line, linear approximations in thin line, asymptotes in dashed line.*

2.3.2 Classification using linearised equation

An interesting further use of the linearisation about a fixed point is that it can also be employed to classify the corresponding fixed point. Consider again an equation $\frac{dy}{dx} = X(y)$ and let $y = a$ be a fixed point.

The linearisation equation:

$$\frac{dz}{dx} = zX'(a). \tag{2.9}$$

can be written in terms of the variable y as

$$\frac{dy}{dx} = (y - a)X'(a). \tag{2.10}$$

As we know, classifying the fixed point a means looking at the sign of $\frac{dy}{dx}$ in the regions $y > a$ and $y < a$. After linearisation, this sign is determined by the term $(y - a)X'(a)$.

- If $X'(a) > 0$, then $\frac{dy}{dx} > 0$ for $y > a$ and $\frac{dy}{dx} < 0$ for $y < a$. Therefore the fixed point $y = a$ is a repellor.

- If $X'(a) < 0$, then $\frac{dy}{dx} > 0$ for $y < a$ and $\frac{dy}{dx} < 0$ for $y > a$. Therefore the fixed point $y = a$ is an attractor.
- If $X'(a) = 0$ this means that linearisation is not a good enough approximation in this case. The nature of the fixed point can not be determined from the linearised equation.

Chapter 3

Solution of Second Order Linear Systems

3.1 Introduction

In Chapter 2 we considered a general 1-dimensional autonomous system which we investigated using the method of linearisation. This enabled us to classify the fixed points using just the linearisation, provided the fixed points were repellors or attractors, and additionally find good approximate solutions to the equation close to the fixed points. For the 1-dimensional system there is very little to say, however the methodology can be extended into higher dimensions. We now look at 2-dimensional systems and as before we will start by considering the linear autonomous systems. First however we review the types of 2-dimensional systems we might consider:

Review of types of 2-dimensional problems

The general form of the 2-dimensional systems we are considering is:

$$\frac{dx_1}{dt} = X_1(x_1, x_2, t) \qquad \frac{dx_2}{dt} = X_2(x_1, x_2, t) \tag{3.1}$$

and using a vector notation we write:

$$\frac{d\underline{x}}{dt} = \underline{X}(\underline{x}, t)$$

where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\underline{X}(\underline{x}, t) = \begin{pmatrix} X_1(x_1, x_2, t) \\ X_2(x_1, x_2, t) \end{pmatrix}$.

Example 3.1

$$\begin{aligned} \frac{dx_1}{dt} &= 33 - 10x_1 - 3x_2 + x_1^2 + t &= X_1(x_1, x_2, t) \\ \frac{dx_2}{dt} &= -18 + 6x_1 + 2x_2 - x_1x_2 + t^2 &= X_2(x_1, x_2, t) \end{aligned}$$

Example 3.2

$$\frac{d^2\theta}{dt^2} = -\sin\theta$$

Setting $x_1 = \theta$ and $x_2 = d\theta/dt$ this equation can be rewritten as:

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 = X_1(x_1, x_2, t)\\ \frac{dx_2}{dt} &= -\sin x_1 = X_2(x_1, x_2, t)\end{aligned}$$

Example 3.3

$$\begin{aligned}\frac{dx_1}{dt} &= 33 - 10x_1 - 3x_2 &= X_1(x_1, x_2)\\ \frac{dx_2}{dt} &= -18 + 6x_1 + 2x_2 &= X_2(x_1, x_2)\end{aligned}$$

In the above we see that in Example 3.1 t is contained explicitly in the righthand side of both the equations whereas in the other two examples t does not appear. We make the following definition:

Definition
The system

$$\frac{d\underline{x}}{dt} = \underline{X}(\underline{x})$$

is said to be **autonomous** since t (*the independent variable*) is not contained explicitly in the right-hand side. Here the variables which are underlined represent vectors. For example $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Note:

- Example 3.1 is non-autonomous and non linear. It is said to be non-linear due to the x_1^2 term and the x_1x_2 term, though just one of these terms would have been sufficient to make the system non-linear.
- Example 3.2 is autonomous and non-linear. The non-linearity is due to the $\sin x_1$ term.
- Example 3.3 is autonomous and linear and can be written in the form

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} -10 & -3 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 33 \\ -18 \end{pmatrix}$$

We make the following definition:

Definition
The system:

$$\frac{d\underline{x}}{dt} = A(t)\underline{x} + \underline{b}(t),$$

where $\underline{x}$ and $\underline{b}$ are in $\mathbb{R}^2$ and $A(t)$ is a 2×2 matrix is called an 2^{nd} **order linear system**

If in addition A and $\underline{b}$ are constant the system is also autonomous. Example 3.3 is an autonomous linear system.

Fixed points

As with the one dimensional system we define a fixed point of the autonomous equations as follows:

Definition
A **fixed point** of the system:

$$\frac{d\underline{x}}{dt} = \underline{X}(\underline{x}) \tag{3.2}$$

is a constant vector $\underline{a}$ such that $\underline{X}(\underline{a}) = \underline{0}$. We note that $\underline{x} = \underline{a}$ is also a solution of Eq. (3.2).

- In Example 3.2 the fixed points occur at $x_2 = 0$ and $-\sin x_1 = 0 \quad \Rightarrow \quad x_1 = n\pi$. Thus the fixed points are on the x_1-axis at multiples of π.
- In Example 3.3 the fixed points are found by solving:

  $$33 - 10x_1 - 3x_2 = 0 \qquad \text{and} \qquad -18 + 6x_1 + 2x_2 = 0$$

  which gives $x_1 = 6$ and $x_2 = -9$. Thus this system has only one fixed point at $(2, -9)$.
- We will not consider fixed points of non-autonomous systems.

3.2 Linear autonomous second order systems

So far in this chapter we have been looking at systems of two first order linear equations. For those systems using a vector notation as above is useful. It turns out that every second order linear differential equation can be re-written as a system of two first order linear equations as follows: consider the general equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0, \tag{3.3}$$

where a, b, c are constants.

If we define $x = x_1$ and $\frac{dx}{dt} = x_2$ then Eq. (3.3) gives:

$$\frac{dx_1}{dt} = x_2 \quad \text{and} \quad \frac{dx_2}{dt} = -\frac{c}{a}x_1 - \frac{b}{a}x_2.$$

Which can be written in the standard $\dot{\underline{x}} = A\underline{x}$ form as:

$$\dot{\underline{x}} = \begin{pmatrix} 0 & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.4}$$

Here $\dot{\underline{x}} = \frac{dx}{dt}$. Thus we can obtain a solution of Eq. (3.4) by solving Eq. (3.3). Rather than do this in general consider the following example

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0.$$

You have learned last year how to solve this kind of equation. We try an exponential type solution $e^{\lambda t}$, substitute into the equation and get an equation for λ. In this case the equation is $\lambda^2 + 3\lambda + 2 = 0$ which gives two solutions $\lambda_1 = -1$ and $\lambda_2 = -2$. Therefore, the general solution to this equation would be $x = \alpha e^{-t} + \beta e^{-2t}$ where α and β are arbitrary constants.

Let us now try to solve the same equation $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$ by rewriting it first in matrix form. As before we define $x_1 = x$ and $x_2 = \dot{x}_1$. We then have equations

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -3x_2 - 2x_1. \tag{3.5}$$

In matrix form, the equation can be written as

$$\dot{\underline{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.6}$$

The new equations (3.5) and their matrix form can now be solved by using properties of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}.$$

The main properties which will play a crucial role are the eigenvalues and eigenvectors of A. The eigenvalues can be found by computing the determinant

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda(3+\lambda) + 2 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2). \tag{3.7}$$

The determinant has zeros at $\lambda_1 = -1$ and $\lambda_2 = -2$. The corresponding eigenvectors associated to these eigenvalues are obtained by solving the equations

$$A\underline{E}_1 = -\underline{E}_1 \qquad A\underline{E}_2 = -2\underline{E}_2. \tag{3.8}$$

Explicitly, for the 1st eigenvalue,

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}, \tag{3.9}$$

which gives equations $b = -a$ and $-2a - 3b = -b$. Both equations are in fact equivalent so that any vector with $a = -b$ is an eigenvector of A. We will choose $a = -b = 1$, although any other choice would also work. This fixes

$$\underline{E}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.10}$$

Performing a similar calculation it is possible to show that

$$\underline{E}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{3.11}$$

You also know from Linear Algebra, that the matrix $P = (\underline{E}_1, \underline{E}_2)$ whose columns are the eigenvectors above, diagonalizes A. That is,

$$P^{-1}AP = J = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}. \tag{3.12}$$

Notice that this also implies $A = PJP^{-1}$. All these Linear Algebra results allow us to transform the equation (3.6) into a simpler equation that we can easily solve[1]. If we write (3.6) as

$$\dot{\underline{x}} = A\underline{x} \quad \Leftrightarrow \quad \dot{\underline{x}} = PJP^{-1}\underline{x}. \tag{3.13}$$

Multiplying both sides of the last equation by P^{-1} we have

$$P^{-1}\dot{\underline{x}} = JP^{-1}\underline{x}, \tag{3.14}$$

which means that we just got a new equation $\dot{\underline{y}} = J\underline{y}$ for the vector $\underline{y} = P^{-1}\underline{x}$. The advantage of this is that the equation for $\underline{y}$ is very simple to solve. In components, it is

$$\dot{\underline{y}} = J\underline{y} \quad \Leftrightarrow \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{3.15}$$

Equivalently we have equations $\dot{y}_1 = -y_1$ and $\dot{y}_2 = -2y_2$. These equations can be easily solved because they are independent from each other, that is the equation for y_1 only involves y_1 and the same for y_2. They have the standard solutions $y_1 = \alpha e^{-t}$ and $y_2 = \beta e^{-2t}$, where α and β are arbitrary constants.

Once we have $\underline{y}$ we can get $\underline{x}$ simply by

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P\underline{y} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha e^{-t} \\ \beta e^{-2t} \end{pmatrix} = \begin{pmatrix} \alpha e^{-t} + \beta e^{-2t} \\ -\alpha e^{-t} - 2\beta e^{-2t} \end{pmatrix}. \tag{3.16}$$

If we look at the solution for $x_1 = \alpha e^{-t} + \beta e^{-2t}$ we see that it is exactly the same solution we obtained above for $x = x_1$ by the usual method. The solution for $x_2 = -\alpha e^{-t} - 2\beta e^{-2t}$ is nothing but $\dot{x}_1$ as we expected from our original definition.

We can further look at the structure of (3.16) and draw some general conclusions. We have

$$\underline{x} = \begin{pmatrix} \alpha e^{-t} + \beta e^{-2t} \\ -\alpha e^{-t} - 2\beta e^{-2t} \end{pmatrix} = \alpha e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \alpha e^{\lambda_1 t}\underline{E}_1 + \beta e^{\lambda_2 t}\underline{E}_2. \tag{3.17}$$

General solution

From the example above we can deduce some general results. First, given a linear equation of the

[1]Recall that, given a general 2×2 matrix

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

its inverse is simply given by

$$P^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

form $\underline{\dot{x}} = A\underline{x}$, its general solution is in general a linear combination of fundamental solutions of the form $e^{\lambda_i t}\underline{E}_i$, with $i = 1, 2$ and $\lambda_i, \underline{E}_i$ being the eigenvalues and eigenvectors associated to the matrix A, respectively. It is quite easy to show that this is true in general.

Let us substitute the solutions $e^{\lambda_i t}\underline{E}_i$ directly onto the equation $\underline{\dot{x}} = A\underline{x}$. We find,

$$\underline{\dot{x}} = A\underline{x} \quad \Rightarrow \quad \frac{de^{\lambda_i t}\underline{E}_i}{dt} = A\left(e^{\lambda_i t}\underline{E}_i\right) \quad \Rightarrow \quad \lambda_i e^{\lambda_i t}\underline{E}_i = e^{\lambda_i t}A\underline{E}_i \quad \Rightarrow \quad \lambda\underline{E}_i = A\underline{E}_i$$

Thus we deduce that if:

$$A\underline{E}_i = \lambda_i\underline{E}_i \tag{3.18}$$

then

$$\underline{x} = e^{\lambda_i t}\underline{E}_i \qquad \text{is a solution of} \qquad \underline{\dot{x}} = A\underline{x}.$$

It can be shown that if $\underline{E}_1$ and $\underline{E}_2$ are two independent solutions of $A\underline{E} = \lambda\underline{E}$, with $\lambda = \lambda_1$ and λ_2 respectively, then the general solution of the 2-dimensional system $\underline{\dot{x}} = A\underline{x}$ is given by:

$$\underline{x} = C_1 e^{\lambda_1 t}\underline{E}_1 + C_2 e^{\lambda_2 t}\underline{E}_2 \tag{3.19}$$

where C_1 and C_2 are arbitrary constants.

Unfortunately it is not always possible to find two vectors, $\underline{E}_1$ and $\underline{E}_2$, that are both independent and real. We therefore need to study the solutions of Eq. (3.18) and their relationship to the solution of $\underline{\dot{x}} = A\underline{x}$ in more detail.

3.3 Linear Algebra - eigenvalues and eigenvectors

We now take time out to review some of the basic results from Linear Algebra that are relevant to our problem. Namely the solutions of Eq. (3.18) and the subsequent idea of reducing A to a Jordan canonical form (in the previous example this would be the matrix J obtained by diagonalising A).

How to work out the eigenvectors and eigenvalues of a matrix should be familiar to you from the Linear Algebra module and I will probably not go through the example below in the class. Questions are welcome in case any difficulties arise.

Predominantly A will be a real constant 2×2 matrix, however most of the definitions and results are applicable to the n-dimensional case.

Definition

Given an $n \times n$ real matrix A then a non-zero vector $\underline{E}$ such that:

$$A\underline{E} = \lambda\underline{E} \tag{3.20}$$

is called an **eigenvector** corresponding to the **eigenvalue** λ. Equation 3.20 is called the **eigenvalue equation** for A.

Note:

- In this definition $\underline{E} \neq \underline{0}$ however λ may be zero.
- From equation 3.19 we see how the eigenvalues and vectors may feature in our solution.

3.3.1 Evaluation of eigenvalues and eigenvectors

Consider the following example:

Given $A = \begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix}$ find $\underline{E} = \begin{pmatrix} a \\ b \end{pmatrix} \neq \underline{0}$ and λ such that $A \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$

Thus we are required to find a, b (not both zero) and λ such that:

$$\begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

In terms of simultaneous equations this gives:

$$\begin{aligned} 4a - 6b = & \ \lambda a \quad \Rightarrow \quad (4-\lambda)a - 6b = \ 0 \\ 3a - 5b = & \ \lambda b \quad \Rightarrow \quad 3a + (-5-\lambda)b = \ 0 \end{aligned}$$

Writing this in matrix form we have;

$$\begin{pmatrix} 4-\lambda & -6 \\ 3 & -5-\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

or even more concisely:

$$(A - \lambda I) \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \tag{3.21}$$

where I is the usual 2×2 identity matrix. To obtain a, b and λ we argue as follows:

- If $(A - \lambda I)$ has an inverse then $\begin{pmatrix} a \\ b \end{pmatrix} = (A - \lambda I)^{-1}\underline{0} = \underline{0}$ Thus the only solution is $a = b = 0$ which is not permissible for an eigenvector.
- Thus we require λ such that $(A - \lambda I)$ has no inverse. The condition that a matrix has no inverse is that its determinant is zero. Thus we require λ such that

$$\det(A - \lambda I) = 0 \tag{3.22}$$

  Equation 3.22 is called the characteristic equation for A.

General result
The eigenvalues of A are given by the roots of the characteristic equation:

$$\det(A - \lambda I) = 0$$

In general when A is $n \times n$ this is an n^{th} degree polynomial equation. In our example, as we see below, it is a quadratic.

To find the eigenvalues
Returning to our problem:

$$\det \begin{pmatrix} 4-\lambda & -6 \\ 3 & -5-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (4-\lambda)(-5-\lambda) - (-6)(3) = 0 \quad \Rightarrow$$

$$\lambda^2 + \lambda - 2 = 0 \quad \Rightarrow \quad \lambda_1 = +1 \text{ and } \lambda_2 = -2$$

labelling the two values of λ as λ_1 and λ_2.

To find the eigenvectors
For each value of λ we need to find a and b (*not both zero*) such that:

$$\begin{pmatrix} 4-\lambda & -6 \\ 3 & -5-\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

$\underline{\lambda_1 = \quad 1}$

$$\begin{pmatrix} 4-1 & -6 \\ 3 & -5-1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow$$

$$\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad \begin{array}{rl} 3a - 6b = & 0 \\ 3a - 6b = & 0 \end{array} \quad \Rightarrow \quad a = 2b$$

Thus corresponding to $\lambda_1 = 1$ we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2b \\ b \end{pmatrix} = b \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = -2}$

$$\begin{pmatrix} 4+2 & -6 \\ 3 & -5+2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow$$

$$\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad \begin{array}{rl} 6a - 6b = & 0 \\ 3a - 3b = & 0 \end{array} \quad \Rightarrow \quad a = b$$

Thus corresponding to $\lambda_2 = -2$ we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In each case we see that we do not obtain a unique eigenvector but linear multiple of a vector. If we draw a line through the origin along the direction of an eigenvector, this is called the eigenline corresponding to the eigenvector. In the above we set

$$\underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \underline{E}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

These eigenvectors alone define two eigenlines (the lines that go by the origin and the points $(2,1)$ and $(1,1)$, respectively) . These eigenlines will play an important role later on when we draw the phase diagrams associated to systems of equations like the one of section 3.2.

3.3.2 Canonical Forms

Let us think again about the example of section 3.2. A crucial step in our solution procedure was the fact that we could diagonalise the matrix A by multiplying it by the matrix $P = (\underline{E}_1, \underline{E}_2)$. When doing this we got a diagonal matrix $J = P^{-1}AP$ in terms of which the system of equations simplified considerably.

A very special feature of the matrix A in section 3.2 is that it had two different, real eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$. Because of this, it also had two linearly independent eigenvectors in terms of which we could construct the matrix P.

In order to be able to solve any systems of two linear equations we need to answer the question: how can we solve such a system if the eigenvalues are equal to each other or non real? How can we construct the matrices P and J in such a case?

The general result goes as follows:

Theorem

Given a 2×2 real matrix A there exists a non-singular real matrix P such that:

$$A = PJP^{-1}$$

where J is one of the four matrices:

- If the eigenvalues of A are λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ then $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
- If the eigenvalues of A are $\lambda_1 = \lambda_2$ with $\lambda_1 \in \mathbb{R}$ then $J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$.
- If the eigenvalues of A are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$ then $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.
- If $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ with λ_1 real, then $J = A$.

J is called the canonical or Jordan Normal form of A.

We will now analyise these four cases in more detail. We need to answer two questions: first, how to find the matrix P that relates A to J in each of the four cases above and second, once we have found P and J, how to solve the resulting system of equations.

Case 1 λ_1 and λ_2 **real** and **different**

As we saw in the example discussed earlier, the matrix P which relates A to its Jordan Normal form J in this case is simply

$$P = (\underline{E}_1, \underline{E}_2), \qquad \text{with} \qquad \underline{E}_1, \underline{E}_2 \qquad \text{eigenvectors of } A, \tag{3.23}$$

and

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{3.24}$$

If we want to solve the equation $\dot{\underline{x}} = A\underline{x}$ we just have to follow the following steps:

$$\dot{\underline{x}} = A\underline{x} \quad \Leftrightarrow \quad \dot{\underline{x}} = PJP^{-1}\underline{x} \quad \Leftrightarrow \quad P^{-1}\dot{\underline{x}} = JP^{-1}\underline{x}. \tag{3.25}$$

Defining now the vector $\underline{y} = P^{-1}\underline{x}$, we see that the original equation $\dot{\underline{x}} = A\underline{x}$ has been transformed into the new equation

$$\dot{\underline{y}} = J\underline{y}, \tag{3.26}$$

which is the Canonical form of the original equation. This equation does not look too different, but it is much easier to solve because J is the diagonal matrix (3.24). We have,

$$\dot{\underline{y}} = J\underline{y} \quad \Leftrightarrow \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \Leftrightarrow \quad \dot{y}_1 = \lambda_1 y_1,\ \dot{y}_2 = \lambda_2 y_2, \tag{3.27}$$

therefore

$$y_1 = C_1 e^{\lambda_1 t} \quad \text{and} \quad y_2 = C_2 e^{\lambda_2 t}, \qquad \text{with } C_1, C_2 \text{ arbitrary constants.} \tag{3.28}$$

This can also be written as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Having now solved the Canonical form of the system of equation we still need to find the solution for $\underline{x}$. From before we have that

$$\underline{x} = P\underline{y} = (\underline{E}_1, \underline{E}_2)\underline{y} = C_1 e^{\lambda_1 t} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_2 t} P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 e^{\lambda_1 t}\underline{E}_1 + C_2 e^{\lambda_2 t}\underline{E}_2. \tag{3.29}$$

Example: Here is a concrete example of a system belonging to case 1. As an exercise work out the solutions below.
Consider the system of equations

$$\dot{x}_1 = 7x_1 - 2x_2, \qquad \dot{x}_2 = 2x_1 + 2x_2.$$

- In matrix form this is $\dot{\underline{x}} = A\underline{x}$ with

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 7 & -2 \\ 2 & 2 \end{pmatrix}.$$

- The matrix A has the following eigenvalues and eigenvectors (check this!):

$$\lambda_1 = 6, \quad \lambda_2 = 3 \qquad \text{and} \quad \underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \underline{E}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

  Notice that if you multiply these eigenvectors by a real number you will still get an eigenvector. Any choice is allowed!

- The matrix P is given by

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and therefore, its inverse is

$$P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

You can check that

$$P^{-1}AP = J = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}.$$

- The vector $\underline{y}$ which solves the system of equations in its Canonical form is

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{6t} \\ C_2 e^{3t} \end{pmatrix} = C_1 e^{6t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which gives the final solution

$$\underline{x} = P\underline{y} = C_1 e^{6t} P\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 e^{6t}\underline{E}_1 + C_2 e^{3t}\underline{E}_2 = C_1 e^{6t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

or, in components,

$$x_1 = 2C_1 e^{6t} + C_2 e^{3t} \qquad x_2 = C_1 e^{6t} + 2C_2 e^{3t}.$$

Case 2 $\lambda_1 = \lambda_2$, one real eigenvalue.

When both eigenvalues are equal it is not possible to find two independent eigenvectors as in the previous case. This means that in order to construct the matrix P we need the eigenvector $\underline{E}_1$ associated to the single eigenvalue λ_1 and one other vector which we will call $\underline{J}_1$. The matrix P is then constructed as before

$$P = (\underline{E}_1, \underline{J}_1), \tag{3.30}$$

where $\underline{J}_1$ is a vector which can be constructed as follows: Linear algebra guarantees that in this case there will exist a vector $\underline{J}_1$ such that

$$(A - \lambda_1 I)\underline{J}_1 = \underline{E}_1. \tag{3.31}$$

Therefore, once we have computed the eigenvector $\underline{E}_1$, we just have to substitute in (3.31) and solve for the vector $\underline{J}_1$. Once this has been done, the solution procedure is very similar to that for case 1. The equations (3.25) and (3.26) hold as before with P given by (3.30) and J given by

$$J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$

This means that the equation for the vector $\underline{y}$ in this case is

$$\dot{\underline{y}} = J\underline{y} \qquad \Leftrightarrow \qquad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

or, in components,

$$\dot{y}_1 = \lambda_1 y_1 + y_2, \qquad \dot{y}_2 = \lambda_1 y_2.$$

The second equation is of the same type as for case 1 and is solved by $y_2 = C_2 e^{\lambda_1 t}$, where C_2 is an arbitrary constant. Substituting into the first equation we have,

$$\dot{y}_1 = \lambda_1 y_1 + C_2 e^{\lambda_1 t},$$

which we can rewrite as

$$\dot{y}_1 - \lambda_1 y_1 = C_2 e^{\lambda_1 t}.$$

This equation can be solved by first solving the homogeneous equation

$$\dot{y}_1 = \lambda_1 y_1 \quad \Rightarrow \quad y_1 = C_1 e^{\lambda_1 t},$$

then finding a particular solution to the full equation. Since $e^{\lambda_1 t}$ is already a solution of the homogenous equation, we must try a solution of the form $\alpha t e^{\lambda_1 t}$. By plugging this into the equation we find that $\alpha = C_2$. Therefore,

$$y_1 = e^{\lambda_1 t}(C_2 t + C_1),$$

where C_1 and C_2 are arbitrary constants. The final solutions are

$$y_1 = (C_1 + tC_2)e^{\lambda_1 t} \quad \text{and} \quad y_2 = C_2 e^{\lambda_1 t}.$$

To obtain the solutions in the original coordinates we have to compute

$$\underline{x} = P\underline{y} = (C_1 + tC_2)e^{\lambda_1 t}\underline{E}_1 + C_2 e^{\lambda_1 t}\underline{J}_1.$$

Example: Consider

$$A = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix}$$

We calculate the eigenvalues and vectors in the usual way,

- $|A - \lambda I| = \begin{vmatrix} 0-\lambda & 4 \\ -1 & 4-\lambda \end{vmatrix} = -\lambda(4-\lambda) + 4 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$
  Thus we have a single eigenvalue $\lambda_1 = 2$.

- Set $\underline{E}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ hence $A\underline{E}_1 = 2\underline{E}_1 \quad \Rightarrow \quad \begin{pmatrix} 0-2 & 4 \\ -1 & 4-2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$
  Thus $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2b \\ b \end{pmatrix} = b\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and we take $\underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

- We now introduce a new vector to form the second axis. Linear algebra guarantees that in this case there will exist a vector $\underline{J}_1$ such that

$$(A - \lambda_1 I)\underline{J}_1 = \underline{E}_1 \tag{3.32}$$

  where I is the identity matrix. To calculate $\underline{J}_1$ we substitute $\underline{J}_1 = \begin{pmatrix} m \\ n \end{pmatrix}$ into Eq. (3.32) to give:

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad m = 2n - 1$$

  Thus

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 2n-1 \\ n \end{pmatrix} = n\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

  Again we have an infinite choice of vector, thus we make the simplest choice with $n = 0$ and take $\underline{J}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Any other value of n would have been equally valid.

- The matrix P is given by:

$$P = (\underline{E}_1, \underline{J}_1) = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$$

  and it relates the matrix A above to its Jordan form

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = PAP^{-1}.$$

Following the general construction above, the solution for the vector $\underline{y}$ is

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (C_1 + C_2 t)e^{2t} \\ C_2 e^{2t} \end{pmatrix} = (C_1 + C_2 t)e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and therefore, the solutions for $\underline{x} = P\underline{y}$ are

$$\underline{x} = (C_1 + C_2 t)e^{2t} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (C_1 + C_2 t)e^{2t} \underline{E}_1 + C_2 e^{2t} \underline{J}_1,$$

or, in components,

$$x_1 = 2(C_1 + C_2 t)e^{2t} - C_2 e^{2t} \qquad \text{and} \qquad x_2 = (C_1 + C_2 t)e^{2t}.$$

Case 3 λ_1 and λ_2 complex

The eigenvalues of A are complex if when solving the characteristic polynomial the roots of the quadratic are complex.

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$

Without loss of generality we will always assume that $\beta > 0$.

In general, for the $n \times n$ case, provided A is real any complex roots of the characteristic equation will appear in conjugate pairs, as above.

Furthermore since A is real:

$$\text{If} \quad A\underline{E}_1 = \lambda_1 \underline{E}_1 \quad \Rightarrow \quad \overline{A\underline{E}_1} = \overline{\lambda_1 \underline{E}_1} \quad \Rightarrow \quad A(\bar{\underline{E}}_1) = \bar{\lambda}_1 \bar{\underline{E}}_1$$

Where the *over bar* represents the complex conjugate. Thus we see that not only do the eigenvalues occur in conjugate pairs but so do the corresponding eigenvectors.
ie if $(\lambda_1, \underline{E}_1)$ is an eigenvalue/vector pair then so is $(\bar{\lambda}_1, \bar{\underline{E}}_1)$.

As in case 1, we could use the eigenvectors $\underline{E}_1$ and $\underline{E}_2$ to construct the matrix P that transforms A to its Jordan Normal form. However, the eigenvectors $\underline{E}_1$ and $\underline{E}_2$ are now complex, which would lead to a complex matrix P. We are interested however in working always with real quantities. Therefore to avoid this problem we can instead choose a different pair of vectors, which is real to construct our matrix P. The vectors we will use are not $\underline{E}_1$ and $\underline{E}_2$ but the real and imaginary parts of $\underline{E}_1$ instead. It turns out that the real and imaginary parts of $\underline{E}_1$ are real vectors and furthermore it can be shown that they are independent.

We define,

$$\underline{e}_1 = \mathrm{Re}(\underline{E}_1) \quad \text{and} \quad \underline{e}_2 = \mathrm{Im}(\underline{E}_1)$$

$\underline{e}_1$ and $\underline{e}_2$ are used now to construct the matrix

$$P = (\underline{e}_1, \underline{e}_2)$$

which transforms A to its Jordan Normal form

$$J = PAP^{-1} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \tag{3.33}$$

and performs the change of coordinates by the usual formula $\underline{x} = P\underline{y}$.
We now have to use these results to solve the equation $\dot{\underline{x}} = A\underline{x}$, when A has complex conjugated eigenvalues. In its canonical form the equation becomes,

$$\dot{\underline{y}} = J\underline{y} \qquad \Leftrightarrow \qquad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \Rightarrow \qquad \begin{array}{rcr} \dot{y}_1 & = & \alpha y_1 + \beta y_2 \\ \dot{y}_2 & = & -\beta y_1 + \alpha y_2 \end{array}$$

To solve these equations we transform to polar coordinates using the substitutions
$y_1 = r\cos\theta$ and $y_2 = r\sin\theta$ with $r \geq 0$ to obtain:

$$\begin{array}{rcccll} \dot{y}_1 & = & \dot{r}\cos\theta - r\dot{\theta}\sin\theta & = & \alpha r\cos\theta + \beta r\sin\theta & \quad (i) \\ \dot{y}_2 & = & \dot{r}\sin\theta + r\dot{\theta}\cos\theta & = & -\beta r\cos\theta + \alpha r\sin\theta & \quad (ii) \end{array}$$

To eliminate $\dot{\theta}$ we form $(i) \times \cos\theta$ + $(ii) \times \sin\theta$ to give:

$$\dot{r} = \alpha r \quad \Rightarrow \quad r = r_0 e^{\alpha t} \qquad\qquad \text{where } r_0 \text{ is an arbitrary constant}$$

Similarly $(i) \times \sin\theta$ - $(ii) \times \cos\theta$ gives:

$$-r\dot{\theta} = \beta r \quad \Rightarrow \quad \dot{\theta} = -\beta \quad \Rightarrow \quad \theta = -\beta t + \theta_0 \qquad \text{where } \theta_0 \text{ is an arbitrary constant}$$

Thus

$$y_1 = r_0 e^{\alpha t}\cos(-\beta t + \theta_0) \qquad y_2 = r_0 e^{\alpha t}\sin(-\beta t + \theta_0), \tag{3.34}$$

or, in vector form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = r_0 e^{\alpha t}\cos(-\beta t + \theta_0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{\alpha t}\sin(-\beta t + \theta_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus

$$\underline{x} = P\underline{y} = (\underline{e}_1, \underline{e}_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1\underline{e}_1 + y_2\underline{e}_2 = r_0 e^{\alpha t}\cos(-\beta t + \theta_0)P\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{\alpha t}\sin(-\beta t + \theta_0)P\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= r_0 e^{\alpha t}\{\, \cos(-\beta t + \theta_0)\underline{e}_1 + \sin(-\beta t + \theta_0)\underline{e}_2 \,\}$$

Example: Given $A = \begin{pmatrix} 6 & -13 \\ 2 & -4 \end{pmatrix}$ find the eigenvalues of A and hence write down its canonical form. Find the eigenvectors of A and hence construct a non-singular matrix P such that Eq. (3.33) holds.

- We first find the eigenvalues in the usual way

$$|A-\lambda I| = \begin{vmatrix} 6-\lambda & -13 \\ 2 & -4-\lambda \end{vmatrix} = (6-\lambda)(-4-\lambda)+26 = \lambda^2-2\lambda+2 = 0$$

  Solving this quadratic gives $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$, hence $\alpha = 1$ and $\beta = 1$ and the canonical form is given by:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

- With $\lambda_1 = 1+i$ we now compute the corresponding eigenvector. Let $\underline{E}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ the eigenvector is given by:

$$\begin{pmatrix} 6-(1+i) & -13 \\ 2 & -4-(1+i) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad \begin{array}{rcll} (5-i)a-13b & = & 0 & (i) \\ 2a-(5+i)b & = & 0 & (ii) \end{array}$$

  The two equation can be shown to be dependent with $(5+i)\times(i) = 13\times(ii)$, thus we can use either (i) or (ii) to form our eigenvector.

$$\text{From } (ii) \qquad \frac{a}{b} = \frac{5+i}{2} \quad \Rightarrow \quad \underline{E}_1 = \begin{pmatrix} 5+i \\ 2 \end{pmatrix}$$

  where we have made the most straightforward choice of a and b to give a value of $\underline{E}_1$. As usual any non-zero multiple of this $\underline{E}_1$ would have been just as good. The other eigenvector would just be the complex conjugate of $\underline{E}_1$, that is

$$\underline{E}_2 = \overline{\underline{E}}_1 = \begin{pmatrix} 5-i \\ 2 \end{pmatrix},$$

  and we do not need to compute it.

- In order to construct the matrix P we have to obtain the two vectors given by the real and imaginary parts of $\underline{E}_1$, that is

$$\underline{e}_1 = \mathrm{Re}\begin{pmatrix} 5+i \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{and} \quad \underline{e}_2 = \mathrm{Im}\begin{pmatrix} 5+i \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

  Thus $P = \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix}$ and by direct calculation you can verify that :

$$A = \begin{pmatrix} 6 & -13 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix}^{-1}$$

- So far we have found,

$$A = \begin{pmatrix} 6 & -13 \\ 2 & -4 \end{pmatrix}, \quad \alpha = 1, \quad \beta = 1, \quad \underline{e}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

therefore, following the general steps above, the solution of $\dot{\underline{x}} = A\underline{x}$ is given by

$$\underline{x} = r_0 e^t \left\{ \cos(-t+\theta_0) \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \sin(-t+\theta_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

In terms of components this breaks down into:

$$x_1 = r_0 e^t \{ \, 5\cos(-t+\theta_0) + \sin(-t+\theta_0) \, \} \quad \text{and} \quad x_2 = 2r_0 e^t \cos(-t+\theta_0)$$

Case 4 $\quad A = \lambda_1 I$

This is a trivial case since if A is a multiple of the identity then the equation $\dot{\underline{x}} = A\underline{x}$ can be solved directly and the matrix A can be considered to be already in canonical form. The equation to be solved is

$$\dot{\underline{x}} = A\underline{x} \quad \Leftrightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which in components becomes

$$\dot{x}_1 = \lambda_1 x_1, \qquad \dot{x}_2 = \lambda_1 x_2,$$

which are solved by

$$x_1 = C_1 e^{\lambda_1 t} \quad \text{and} \quad x_2 = C_2 e^{\lambda_1 t}. \tag{3.35}$$

Chapter 4

Phase Diagrams for Second Order Linear Systems

4.1 Simple fixed points

This section deals with the classification of fixed points of the linear system $\dot{\underline{x}} = A\underline{x} + \underline{b}$ and the construction of the phase diagrams about the fixed points. For most of the section, we will be concentrating on the case when the vector $\underline{b} = 0$ (as we did in the previous chapter).

Recall that for one-dimensional systems (or systems of one equation) there were only three types of fixed points: attractors, repellors and shunts. For systems of two equations there are many more possibilities, as the phase diagram is two dimensional. There are in fact *ten* different types of fixed points which we will be classifying and studying in this chapter!

The fixed points of the system are given by the solutions of the equation $A\underline{x}+\underline{b} = \underline{0}$. If A^{-1} exists, then there exists a unique fixed point given by $\underline{x} = -A^{-1}\underline{b}$. Thus, if $\underline{b} = 0$ the fixed point is at the origin.

We will restrict our attention to the case where A^{-1} exists, ie A is non-singular. We already know that A^{-1} exists if and only if $\det(A) \neq 0$, however we now show that this is equivalent to A not having a zero eigenvalue.
Consider the characteristic polynomial $\det(A - \lambda I)$ and assume that λ_1 and λ_2 are the eigenvalues of A. (λ_1 *and* λ_2 *may be real and different, equal or complex*) Thus we can write:

$$\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Since this equality is true for all λ, setting $\lambda = 0$ gives the result that $\det(A) = \lambda_1\lambda_2$. Thus $\det(A) \neq 0$ if and only if neither λ_1 nor λ_2 are equal to zero. Since we are only going to consider the case when A is non-singular we make the following definition:

Definition
$\underline{x} = \underline{a}$ is a **simple fixed point** of $\dot{\underline{x}} = A\underline{x} + \underline{b}$ if:

- it is a solution of the equation $A\underline{a} + \underline{b} = 0$
- A is non-singular. *Equivalently this is* $\det(A) \neq 0$, A^{-1} *exists, no eigenvalue of* A *is zero.*

  The fixed point is unique and given by $\underline{a} = -A^{-1}\underline{b}$

In the case $\underline{b} = 0$, that is to say if the equation is $\underline{\dot{x}} = A\underline{x}$, the only simple fixed point is at the origin.

Initially we restrict our attention to the problem $\underline{\dot{x}} = A\underline{x}$ and then later look at the more general problem with $\underline{b} \neq 0$.

4.2 Classification of Simple Fixed Points

Before developing the phase diagrams we first write down the classifications of the simple fixed point of the system $\underline{\dot{x}} = A\underline{x} + \underline{b}$. These are listed according to the eigenvalues of A.

1. Eigenvalues real and different
   - **saddle** λ_1 and λ_2 different signs: take $\lambda_1 > 0 > \lambda_2$
   - **unstable node** λ_1 and λ_2 both positive: take $\lambda_1 > \lambda_2 > 0$
   - **stable node** λ_1 and λ_2 both negative: take $0 > \lambda_2 > \lambda_1$
2. Single eigenvalue ($\lambda_1 = \lambda_2$)
   - **unstable improper node** $\lambda_1 > 0$
   - **stable improper node** $\lambda_1 < 0$
3. Complex eigenvalues $\alpha \pm i\beta$ with $\beta > 0$
   - **unstable focus** $\alpha > 0$
   - **stable focus** $\alpha < 0$
   - **centre** $\alpha = 0$
4. $A = \lambda_1 I$
   - **unstable star node** $\lambda_1 > 0$
   - **stable star node** $\lambda_1 < 0$

4.3 Phase Diagram Type l: $\lambda_1 \neq \lambda_2$, real and non-zero

In order to obtain the phase diagram for $\underline{\dot{x}} = A\underline{x}$ we first obtain the phase diagram for the canonical system $\underline{\dot{y}} = J\underline{y}$. In this case the canonical form is given by

$$\underline{\dot{y}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \underline{y} \quad \Rightarrow \quad y_1 = C_1 e^{\lambda_1 t} \quad y_2 = C_2 e^{\lambda_2 t} \quad \Rightarrow \quad y_2 = C y_1^{\lambda_2/\lambda_1} \tag{4.1}$$

The phase diagram, although given by this expression, will depend on the signs of λ_1 and λ_2. Thus we consider the three possible scenarios, namely where the λ's are both positive, both negative or of different sign.

4.3.1 $\lambda_1 > 0 > \lambda_2$: saddle

From Eq 4.1 we can deduce that with these values of lambda:

- $\lambda_2 < 0 \quad \Rightarrow \quad y_2 \to 0 \quad \text{as} \quad t \to \infty$
- $\lambda_2 < 0 \quad \Rightarrow \quad y_2 \to \infty \quad \text{as} \quad t \to -\infty$
- $\lambda_1 > 0 \quad \Rightarrow \quad y_1 \to 0 \quad \text{as} \quad t \to -\infty$
- $\lambda_1 > 0 \quad \Rightarrow \quad y_1 \to \infty \quad \text{as} \quad t \to \infty$
- If $C_1 = 0$ then the trajectory is $y_1 = 0$, that is the y_2-axis is part of the phase diagram. In this case $y_2 = C_2 e^{\lambda_2 t}$ which means that $|y_2|$ grows as as t tends to minus infinity and viceversa (this determines the direction of the arrows in the phase diagram).
- If $C_2 = 0$ then the trajectory is $y_2 = 0$, that is the y_1-axis is also part of the phase diagram, as we can see below. In this case $y_1 = C_1 e^{\lambda_1 t}$, that is to say $|y_1|$ tending to infinity as t tends to infinity.

The phase diagram for the canonical case is seen in Fig. 4.1(a). To find the phase diagram for the full system $\dot{\underline{x}} = A\underline{x}$ we recall that $\underline{x} = P\underline{y}$ where P is the matrix whose columns are the eigenvectors of A.

If $\underline{E}_1$ and $\underline{E}_2$ are the eigenvectors of A corresponding to the eigenvalues λ_1 and λ_2 respectively then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \underline{E}_1 & \underline{E}_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Thus we see that the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ along the y_1-axis is transformed to $\underline{E}_1$ in the x_1-x_2 plane. That is to say the y_1 axis is transformed into the line of $\underline{E}_1$. Similarly the y_2-axis is transformed into the line of $\underline{E}_2$. Thus we see that the diagram is deformed, squashing or stretching the trajectories. Under

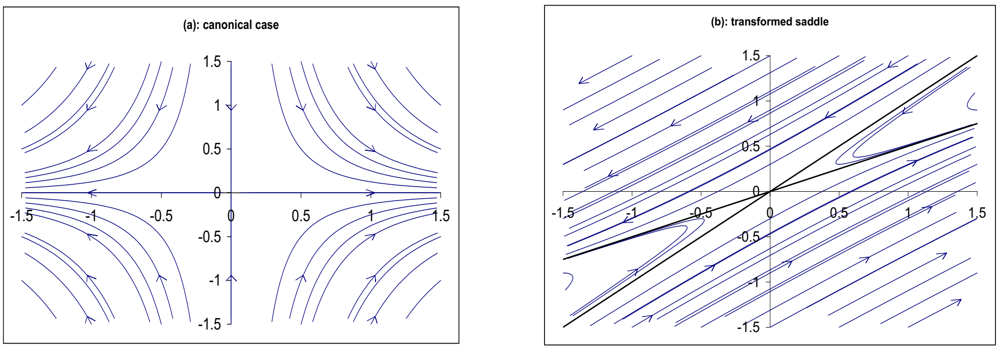


Figure 4.1: (a) canonical saddle with $\lambda_1 = 1$ and $\lambda_2 = -2$. (b) the transformed saddle to give the phase diagram for $\dot{\underline{x}} = A\underline{x}$

this transformation the direction of the arrows on the y_1-y_2 axes is maintained along the lines of $\underline{E}_1$

and $\underline{E}_2$. This is a useful way of determining the direction of the arrows for the whole diagram. The diagram has been drawn using the example of Section 3.3.1 where $\lambda_1 = 1$, $\lambda_2 = -2$, $\underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underline{E}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In Fig.4.1(b) the heavy lines are the lines of the eigenvectors $\underline{E}_1$ and $\underline{E}_2$ respectively.

4.3.2 $\lambda_1 > \lambda_2 > 0$: **unstable node**

From Eq. 4.1 we can deduce

$$\frac{dy_2}{dy_1} = \frac{\frac{dy_2}{dt}}{\frac{dy_1}{dt}} = \frac{\lambda_2 C_2 e^{\lambda_2 t}}{\lambda_1 C_1 e^{\lambda_1 t}} = \frac{\lambda_2}{\lambda_1} C e^{-(\lambda_1 - \lambda_2)t}$$

where C_1 and C_2 have been combined to give C. Hence:

- As $t \to \infty$ $\frac{dy_2}{dy_1} \to 0$ thus the trajectories become horizontal.
- As $t \to -\infty$ $|\frac{dy_2}{dy_1}| \to \infty$ thus the trajectories become vertical.
- Both y_1 and y_2 tend to zero as t tends to minus infinity. (*from* Eq. 4.1)
- Both $|y_1|$ and $|y_2|$ tend to infinity at t tends to plus infinity.(*from* Eq. 4.1)

Additionally from Eq. 4.1 we can see that both the axes are trajectories with arrows pointing away from the origin as follows:

- If $C_1 = 0$ then the trajectory is $y_1 = 0$ and $y_2 = C_2 e^{\lambda_2 t}$, that is to say the y_2-axis with $|y_2|$ tending to zero as t tends to infinity.
- If $C_2 = 0$ then the trajectory is $y_2 = 0$ and $y_1 = C_1 e^{\lambda_1 t}$, that is to say the y_1-axis with $|y_1|$ tending to infinity as t tends to infinity.

Fig. 4.2 is drawn using the data from example 4.1 where the eigenvalues are +6 and +3 and the eigenvectors are $\underline{E}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underline{E}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The lines along these direction are shown in fig. 4.2(b) in bold and are respectively the transformations of the y_1 and y_2 axes.

Example 4.1

Given

$$\dot{x} = 7x_1 - 2x_2 \qquad \dot{x}_2 = 2x_1 + 2x_2$$

- Locate and classify the fixed point.
- Sketch the phase diagram.
- Obtain the solution that passes through the point $(1, 1)$ at $t = 0$.

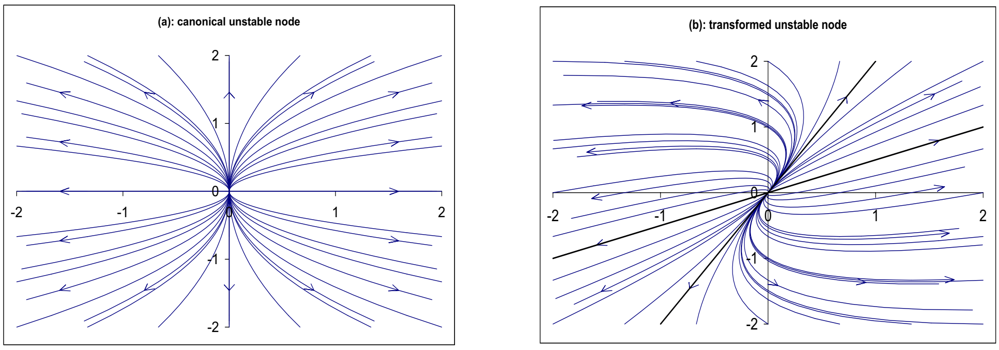


Figure 4.2: (a) canonical unstable node with $\lambda_1 = 6$ and $\lambda_2 = 3$. (b) the transformed unstable node to give the phase diagram for $\underline{\dot{x}} = A\underline{x}$

This is in fact the same example we did in page 37, so we already know the fixed points, eigenvectors and eigenvalues. We will take the solutions we found there and concentrate largely on drawing the phase diagram. Recall the solutions:

$$\underline{x} = C_1 e^{\lambda_1 t}\underline{E}_1 + C_2 e^{\lambda_2 t}\underline{E}_2 = C_1 e^{6t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Or in terms of components:

$$x_1 = 2C_1 e^{6t} + C_2 e^{3t} \qquad x_2 = C_1 e^{6t} + 2C_2 e^{3t}$$

We are given the initial condition that at $t = 0$, $x_1 = x_2 = 1$, which allows us to fix the constants as

$$1 = 2C_1 + C_2 \qquad \text{and} \qquad 1 = C_1 + 2C_2 \quad \Rightarrow \quad C_1 = C_2 = 1/3$$

Thus the solutions are:

$$x_1 = \frac{1}{3}\left(2e^{6t} + e^{3t}\right) \qquad x_2 = \frac{1}{3}\left(e^{6t} + 2e^{3t}\right).$$

The phase diagram is drawn in Fig. 4.2(b) The important features are that the trajectories become parallel to the image of the y_1-axis, that is to say parallel to $\underline{E}_1$ as $t \to \infty$ and appear to emanate from the origin parallel to the image of the y_2-axis, that is to say parallel to $\underline{E}_2$.

4.3.3 $\lambda_1 < \lambda_2 < 0$: stable node

Provided we arrange for $|\lambda_1| > |\lambda_2|$, that is to say λ_1 is now the most negative eigenvalue rather than the most positive, then the phase diagram for the stable node is exactly the same as the one for the unstable node but with the arrows reversed.

4.4 Phase Diagram Type II: $\lambda_1 = \lambda_2 \neq 0$

This is the case where the quadratic characteristic equation has only a single or repeated root which leads to a single independent eigenvector. This type does not include the case where $A = \lambda I$ since in this case any vector is an eigenvector and the problem is virtually trivial; see Type lV. As in Type l. we consider the canonical case first with $\lambda_1 > 0$ and then transform to obtain the phase diagram for $\underline{\dot{x}} = A\underline{x}$.

4.4.1 $\lambda_1 = \lambda_2 > 0$: unstable improper node

The canonical form is given by

$$\underline{\dot{y}} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \underline{y} \quad \Rightarrow \quad y_1 = e^{\lambda_1 t}(C_1 t + C_2) \quad y_2 = C_1 e^{\lambda_1 t} \tag{4.2}$$

The most useful thing we can do is to calculate the gradient of the phase paths in terms of, initially t and then in terms of y_1 and y_2 as follows:

- Differentiating y_1 and y_2 from Eq. (4.2) with respect to t gives
$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{C_1\lambda_1}{C_1 + \lambda_1 C_1 t + \lambda_1 C_2}$$
Thus as $t \to \infty$, $\dfrac{dy_2}{dy_1} \to 0$ and the trajectories become horizontal. Similarly as $t \to -\infty$, $\dfrac{dy_2}{dy_1} \to 0$ and again the trajectories are horizontal. This is in contrast to the case of the unstable node where the trajectories became vertical as $t \to -\infty$

- Directly from the canonical equations in (4.2) $\dot{y}_1 = \lambda_1 y_1 + y_2$ and $\dot{y}_2 = \lambda_1 y_2$ thus:
$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{\lambda_1 y_2}{\lambda_1 y_1 + y_2}$$
Hence the trajectories are vertical along the line $y_2 = -\lambda_1 y_1$. Note that in the unstable case this line has negative gradient since $\lambda_1 > 0$ and is shown in dots in Fig 4.3(a). The line is useful when sketching the diagram however transforming it into the x_1-x_2 phase space does not give a line on which the trajectories are still vertical. To locate such a line in the x_1-x_2 phase space it is necessary to refer back to the original problem and draw the line $\dot{x}_1 = 0$.

- Finally we note from Eq 4.2 that as $t \to -\infty$ the trajectories tend to the origin, since both y_1 and y_2 tend to zero, and as $t \to \infty$ both $|y_1|$ and $|y_2|$ tend to infinity.

Example 4.2

Fig. 4.3 is obtained using the following example:

Given
$$\underline{\dot{x}} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \underline{x}$$
obtain the phase diagram and the general solution.

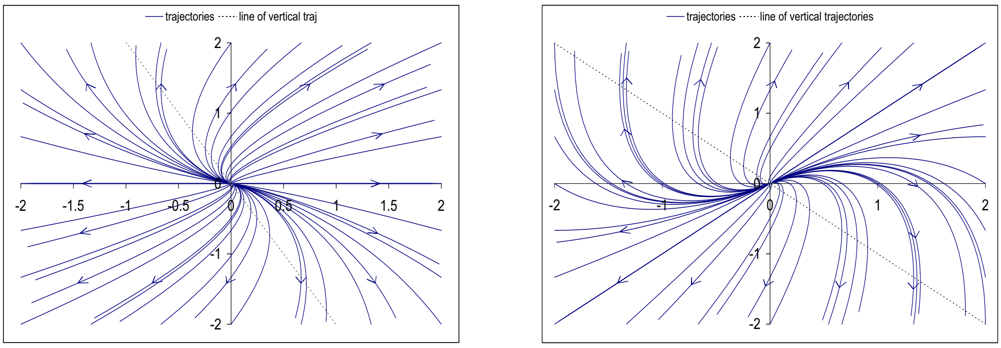


Figure 4.3: (a) canonical unstable improper node with $\lambda_1 = 2$; the line of vertical trajectories is given by $2y_1 + y_2 = 0$. (b) the transformed diagram. The line of vertical trajectories now obtained from $\dot{x} = 0$ which gives $x_1 + x_2 = 0$ (*see example*)

- First calculating the eigenvalues we obtain:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 2$$

- $\underline{\lambda = 2}$ The eigenvector is given by:

$$\begin{pmatrix} 1-2 & 1 \\ -1 & 3-2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad a = b \quad \Rightarrow \quad \underline{E_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- The $\underline{J}_1$ vector, which is required for the solution but not for the diagram is given by:

$$(A - \lambda I)\underline{J_1} = \underline{E_1} \quad \Rightarrow \quad \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad a = b - 1$$

$$\text{Thus } \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b-1 \\ b \end{pmatrix} = b\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

We can take any value of b to give a $\underline{J_1}$, thus we take the simplest value, namely $b = 0$ to give $\underline{J_1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

- The phase diagram is shown in Fig 4.3(b) where we note that the trajectories leave the origin with increasing t since the node is unstable. Additionally as the node is improper the trajectories are parallel to $\underline{E_1}$ both as they leave the origin and as they tend to infinity. ie they tend to become parallel to $\underline{E_1}$ as $|t| \to \infty$. To help sketch the diagram we have plotted in dotted format the line along which the trajectories are vertical. This is given by setting $\dot{x}_1 = 0$ which in this example gives $x_1 + x_2 = 0$.

- Using the vectors $\underline{E}_1$ and $\underline{J}_1$ and the eigenvalue $\lambda = 2$ the solution is given by

$$\underline{x} = e^{2t}(C_1 t + C_2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_1 e^{2t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

or in terms of components:

$$x_1 = e^{2t}(C_1 t + C_2 - C_1) \quad \text{and} \quad x_2 = e^{2t}(C_1 t + C_2)$$

4.4.2 $\lambda_1 = \lambda_2 < 0$: stable improper node

Unfortunately a simple reversal of arrows is not sufficient to describe the difference between the unstable and stable case. If $\lambda_1 < 0$ then the gradient of the dotted line along which the trajectories are vertical in the canonical diagram is now positive. In practice for the improper node it is best to work with the final picture and draw in the line along which $\dot{x}_1 = 0$, draw in the eigenvector $\underline{E_1}$ and add the arrows according to whether the node is stable (*towards the origin*) or unstable (*away from the origin*).

4.5 Phase Diagram Type lll: λ_1 and λ_2 complex $\alpha \pm i\beta$

The canonical equation for this type is given by

$$\dot{\underline{y}} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\underline{y} \qquad \beta > 0$$

As we saw in Eq.3.34 the solution is given by

$$y_1 = r\cos\theta = r_0 e^{\alpha t}\cos(-\beta t + \theta_0) \quad \text{and} \quad y_2 = r\sin\theta = r_0 e^{\alpha t}\sin(-\beta t + \theta_0) \tag{4.3}$$

A simple way to see the shape of the phase space trajectories in this case is to notice that

$$y_1^2 + y_2^2 = r_0^2 e^{2\alpha t}.$$

- If $\alpha = 0$, this is the equation of a circle of radius r_0. The phase space trajectories are concentric circles, centered at the origin (see later in section 4.5.3).
- If $\alpha > 0$, then the equation is not that of a circle anymore, but we can think of it as a "circle" whose radius increases exponentially with time. In the phase diagram the trajectories will look like curves that spiral out of the origin and go to infinity as $t \to \infty$.
- If $\alpha < 0$, then the equation is not that of a circle either, but we can think of it as a "circle" whose radius decreases exponentially with time. In the phase diagram the trajectories will look like curves that spiral in towards the origin and go to infinity as $t \to -\infty$.

Although these equations represent simple curves in the canonical picture the transformation to the x_1-x_2 plane will distort and may change the orientation of the solutions. As we see below, it is essential that we track any change in orientation and try to figure out how the trajectories may get distorted in the original coordinates.

4.5.1 $\alpha > 0$: unstable focus

From the polar coordinates it is clear that with positive α, as t increases, r, the distance from the origin, increases exponentially, whereas θ decreases linearly with time. This mean that as t increase a point P will move away from the origin and rotate clockwise about the origin. This is illustrated in Fig 4.4(a). Looking at the transformation matrix to transform the canonical picture to the x_1-x_2 plane is not very fruitful and the only thing we can usefully do is to check the direction of the spiral in the x_1-x_2 plane by looking at the sign of $\dot{x}_1$ when $x_1 = 0$, that is to say where the trajectories cut the x_2-axis. This is illustrated in Fig 4.4(b) which is constructed from the example below.

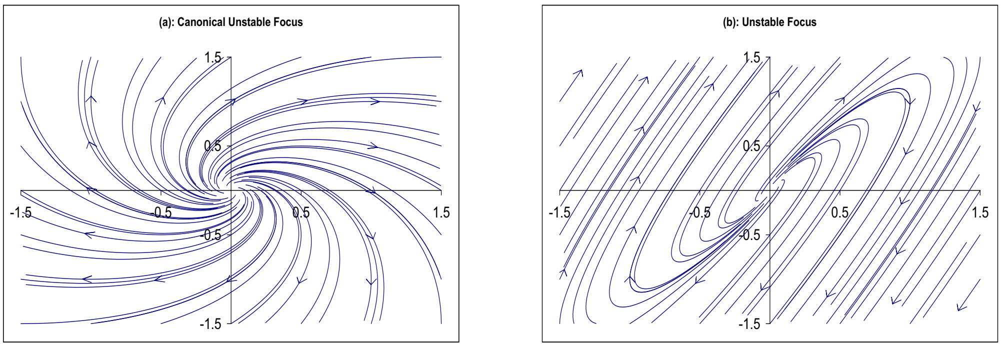


Figure 4.4: (a) Canonical unstable focus with $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ (b) the transformed diagram.

Example 4.3

Fig. 4.4 is obtained using the following example:

Given

$$\dot{\underline{x}} = \begin{pmatrix} -2 & 2 \\ -5 & 4 \end{pmatrix} \underline{x}$$

obtain the phase diagram and the general solution.

- Eigenvalues are found by solving the characteristic equation:

$$\begin{vmatrix} -2-\lambda & 2 \\ -5 & 4-\lambda \end{vmatrix} = (-2-\lambda)(4-\lambda) + 10 = \lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = 1 + i \quad \lambda_2 = 1 - i$$

  Thus the fixed point is an unstable focus.

- The eigenvector is given by

$$\begin{pmatrix} -2-1-i & 2 \\ -5 & 4-1-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \quad \Rightarrow \quad (-3-i)a + 2b = 0$$

$$\Rightarrow \quad \underline{E_1} = \begin{pmatrix} 2 \\ 3+i \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i = \underline{e_1} + \underline{e_2} i$$

- The solution is given by:

$$\underline{x} = r_0 e^t \left(\cos(-t+\theta_0)\ \underline{e_1} + \sin(-t+\theta_0)\ \underline{e_2}\right) = r_0 e^t \left(\cos(-t+\theta_0) \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \sin(-t+\theta_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

  In components this gives:

$$x_1 = 2r_0 e^t \cos(-t+\theta_0) \qquad x_2 = r_0 e^t \left(3\cos(-t+\theta_0) + \sin(-t+\theta_0)\right)$$

- The phase diagram consists of spirals moving away from the origin since $\alpha = 1 > 0$. To check the direction of the spirals we observe the direction of the trajectories as they cross the x_2-axis by considering the sign of $\dot{x}_1$ at $x_1 = 0$. In this example $\dot{x}_1 = -2x_1 + 2x_2$ thus on the x_2-axis, $x_1 = 0$ and $\dot{x}_1 = 2x_2$, which is positive if $x_2 > 0$. Thus the trajectories crossing the positive x_2-axis move in the direction of increasing x_1 and hence are moving in a clockwise direction. See Fig. 4.4(b).

4.5.2 $\alpha < 0$: stable focus

In the canonical picture the only change is that the spiral now tend towards the origin rather than away, however the diagram will need to be redrawn since the rotation is still clockwise. In the x_1-x_2 plane the trajectories still spiral into the origin however as before it will be necessary to check the orientation. (*clockwise or anti-clockwise*)

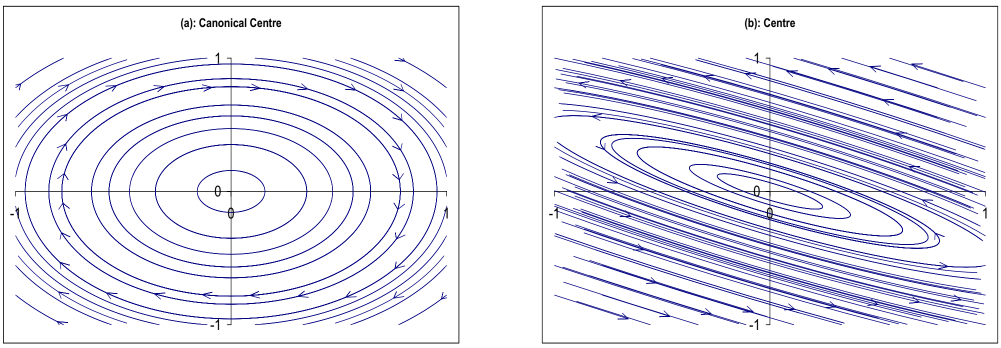


Figure 4.5: (a) Canonical centre (b) Centre for the system $\underline{\dot{x}} = A\underline{x}$ (*see example below*)

4.5.3 $\alpha = 0$: centre

With $\alpha = 0$ it is clear from the solution to the canonical problem (Eq 4.3) that $r = r_0$, thus the trajectories are concentric circles. In the canonical case the orientation is clockwise however in the x_1-x_2 plane the orientation may be different and will have to be checked as before using the sign of $\dot{x}_1$. The following examples is a centre and is illustrated in Fig 4.5(b).

$$\underline{\dot{x}} = \begin{pmatrix} -128 & -272 \\ 80 & 128 \end{pmatrix} \underline{x}$$

In the one dimensional case we mentioned the idea of stability and asymptotic stability (*See section 2.2.*) This is an example of a fixed point that is stable but not asymptotically stable. All the other stable fixed points considered so far in this chapter have also been asymptotically stable as they tend to the fixed point as t tend to infinity.

4.6 Phase Diagram Type IV: $A = \lambda_1 I$

This is the most trivial of the cases and really only arises after linearising a nonlinear problem (*see Chapter 5*). From Eq. (3.35) the solution is given by $x_1 = C_1 e^{\lambda_1 t}$ and $x_2 = C_2 e^{\lambda_1 t}$. Thus dividing these two expressions and relabelling the constants we get that the trajectories are given by $x_2 = Cx_1$, with $C = C_2/C_1$. Thus the trajectories are straight lines through the origin pointing away from the origin if $\lambda_1 > 0$ to give an **unstable star node** and pointing into the origin if $\lambda_1 < 0$ to give a **stable star node**.

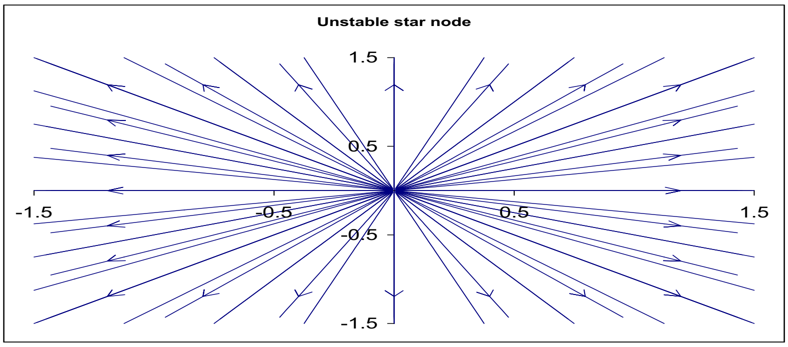


Figure 4.6: Unstable star node; $\dot{\underline{x}} = 2I\underline{x}$

4.7 General Linear System

We conclude this chapter with the solution and phase diagram for the complete linear system $\dot{\underline{x}} = A\underline{x}+\underline{b}$ (so far, we have been looking only at the case $\underline{b} = 0$) . The fixed points of this system are given by the solution of the equation $A\underline{a} + \underline{b} = 0$, however since we are only considering simple fixed points, that is to say A is nonsingular, the fixed point is unique and given by $\underline{a} = -A^{-1}\underline{b}$. If we move the origin to the fixed point using the substitution $\underline{x} = \underline{a} + \underline{z}$ then:

$$\dot{\underline{x}} = A\underline{x} + \underline{b} \quad \Rightarrow \quad \dot{\underline{x}} = \dot{\underline{z}} = A(\underline{a} + \underline{z}) + \underline{b} = A\underline{a} + A\underline{z} + \underline{b}$$

Since $\underline{a}$ is a fixed point $A\underline{a} + \underline{b} = 0$ thus $\dot{\underline{z}} = A\underline{z}$.

Thus the solution and phase diagram for $\dot{\underline{x}} = A\underline{x} + \underline{b}$ is obtained as follows:

- Solve $A\underline{x} + \underline{b} = 0$ to give the fixed point $\underline{a}$

- Solve $\dot{\underline{z}} = A\underline{z}$ for $\underline{z}$
- Write $\underline{x} = \underline{a} + \underline{z}$.
- From Fig. 4.7 we can see that the substitution $\underline{x} = \underline{a} + \underline{z}$ represents a change of origin, thus the phase diagram for $\dot{\underline{x}} = A\underline{x} + \underline{b}$ is obtained by drawing the phase diagram for $\dot{\underline{z}} = A\underline{z}$ about the new origin.

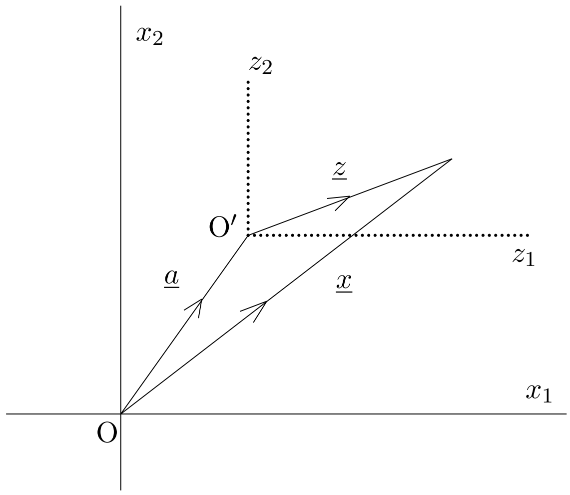


Figure 4.7: Change of origin from O to O ′

Example 4.4

Obtain the solution to :

$$\dot{x}_1 = 4x_1 - 6x_2 - 6 \qquad \dot{x}_2 = 3x_1 - 5x_2 - 4$$

such that $x_1 = 3$ and $x_2 = 4$ at $t = 0$.

Draw the phase diagram for the system.

- Writing the system in the form $\dot{\underline{x}} = A\underline{x} + \underline{b}$ gives:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -6 \\ -4 \end{pmatrix}$$

- The fixed point for the system is given by:

$$\underline{a} = \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix}^{-1} \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -5 & 6 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

  Alternatively you can solve the simultaneous equations $4x_1 - 6x_2 - 6 = 0$ and $3x_1 - 5x_2 - 4 = 0$ by direct elimination.

- The eigenvalues and eigenvectors for this matrix A have already been evaluated in Sec. 3.3.1 to give $\lambda_1 = 1$ and $\lambda_2 = -2$ with $\underline{E_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underline{E_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus the solution of $\dot{\underline{z}} = A\underline{z}$ is given by:

$$\underline{z} = C_1 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the fixed point at $\underline{z} = 0$ is a saddle about this point.

Substituting for $\underline{x} = \underline{a} + \underline{z}$ gives:

$$\underline{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + C_1 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Applying the initial conditions that $x_1 = 3$ and $x_2 = 4$ at $t = 0$ gives:

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 0 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus we have two equations for C_1 and C_2, $2C_1 + C_2 = 0$ and $C_1 + C_2 = 3$. Solving gives $C_1 = -3$ and $C_2 = 6$, hence the particular trajectory through the point $(3, 4)$ is given by:

$$\underline{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 3e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 6e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In terms of components this gives the two solutions:

$$x_1 = 3 - 6e^{t} + 6e^{-2t} \qquad x_2 = 1 - 3e^{t} + 6e^{-2t}$$

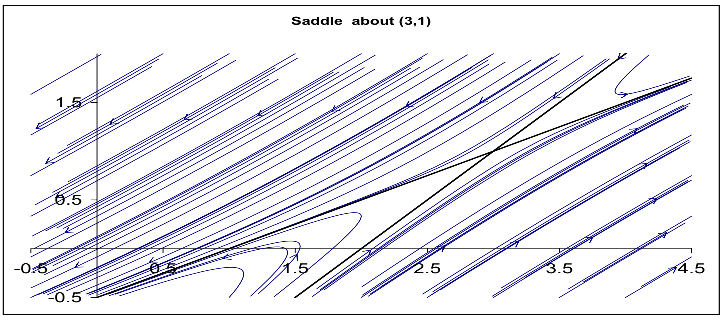


Figure 4.8: Saddle about the fixed point $(3, 1)$ for $\dot{\underline{x}} = A\underline{x} + \underline{b}$

- The phase diagram is given in Fig 4.8. Note that this is a saddle drawn about the fixed point at $(3, 1)$ as origin. The eigenvectors $\underline{E_1}$ and $\underline{E_2}$ are drawn relative to this origin and thus the trajectories move out along the line of $\underline{E_1}$ and in towards this origin along the line of $\underline{E_2}$.

Chapter 5

Nonlinear Second Order Systems

In this chapter we study two dimensional autonomous nonlinear systems of differential equations. In general such systems are not solvable in terms of elementary functions such as exponentials, polynomials and trigonometric functions. The method of approach is to linearise the systems about their fixed points and then construct the phase diagrams accordingly to give overall or global pictures. The linear solutions can also be used to approximate the exact solutions near the fixed points. This approach was seen in the one dimensional case in Section 2.3.

5.1 Linearisation in 2-dimensions

Let $\underline{x} = \underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ be a fixed point of the system $\dot{\underline{x}} = \underline{X}(\underline{x})$. (ie. $\underline{X}(\underline{a}) = \underline{0}$)

Move the origin to the fixed point using the substitution $\underline{x} = \underline{z} + \underline{a}$. (See fig 4.7)

Thus $\dot{\underline{x}} = \dot{\underline{z}} = \underline{X}(\underline{z} + \underline{a})$ or expressing this in coordinates:

$$\dot{z}_1 = X_1(z_1 + a_1, z_2 + a_2) \quad \text{and} \quad \dot{z}_2 = X_2(z_1 + a_1, z_2 + a_2)$$

We now expand the righthand sides of these equations using Taylor's expansion in two variables to give:

$$\begin{aligned}\dot{z}_1 = X_1(a_1, a_2) + z_1\frac{\partial X_1(a_1, a_2)}{\partial x_1} + z_2\frac{\partial X_1(a_1, a_2)}{\partial x_2} + \\ \frac{1}{2!}\left(z_1^2\,\frac{\partial^2 X_1(a_1, a_2)}{\partial x_1^2} + 2z_1z_2\frac{\partial^2 X_1(a_1, a_2)}{\partial x_1\partial x_2} + z_2^2\,\frac{\partial^2 X_1(a_1, a_2)}{\partial x_2^2}\right) + \dots\end{aligned} \tag{5.1}$$

$$\begin{aligned}\dot{z}_2 = X_2(a_1, a_2) + z_1\frac{\partial X_2(a_1, a_2)}{\partial x_1} + z_2\frac{\partial X_2(a_1, a_2)}{\partial x_2} + \\ \frac{1}{2!}\left(z_1^2\,\frac{\partial^2 X_2(a_1, a_2)}{\partial x_1^2} + 2z_1z_2\frac{\partial^2 X_2(a_1, a_2)}{\partial x_1\partial x_2} + z_2^2\,\frac{\partial^2 X_2(a_1, a_2)}{\partial x_2^2}\right) + \dots\end{aligned} \tag{5.2}$$

Since $\underline{x} = \underline{a}$ is a fixed point we have that $\underline{X}_1(a_1, a_2)$ and $\underline{X}_2(a_1, a_2)$ are both zero. If additionally we stay close to $\underline{x} = \underline{a}$ both z_1 and z_2 are small and thus in Eq. (5.1) and Eq. (5.2) we can ignore all terms of second order and above (that is to say z_1^2, z_2^2, z_1z_2 and above). Hence we form the linear approximation as follows:

$$\dot{z}_1 = z_1\frac{\partial X_1(a_1,a_2)}{\partial x_1} + z_2\frac{\partial X_1(a_1,a_2)}{\partial x_2} \tag{5.3}$$

$$\dot{z}_2 = z_1\frac{\partial X_2(a_1,a_2)}{\partial x_1} + z_2\frac{\partial X_2(a_1,a_2)}{\partial x_2} \tag{5.4}$$

This can be written in matrix form as:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_1}{\partial x_2} \\[2ex] \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_2}{\partial x_2} \end{pmatrix}_{(a_1,a_2)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{5.5}$$

The notation (a_1, a_2) in Eq 5.5 indicates that the partial derivatives in the matrix should be evaluated at $x_1 = a_1$ and $x_2 = a_2$.

The matrix in Eq 5.5 expressed at the general point (x_1, x_2) is called the **Jacobian** matrix of $\underline{X}(\underline{x})$.

We now consider an example of a nonlinear system and attempt to construct its global phase diagram by initially constructing the phase diagrams close to its fixed points using the linearised equations about the fixed points.

Example 5.1

Consider the following equations:

$$\dot{x}_1 = x_1 + x_2^2 = X_1 \quad \text{and} \quad \dot{x}_2 = x_1^2 + x_2 = X_2$$

1. **Find the fixed points**
   The fixed points occur when $X_1(x_1, x_2) = X_2(x_1, x_2) = 0$. Thus

   $$x_1 + x_2^2 = 0 \quad (i) \qquad \text{and} \qquad x_1^2 + x_2 = 0 \quad (ii)$$

   Substituting for x_1 from (i) into (ii) gives $x_2^4 + x_2 = 0$ which implies that $x_2 = 0$ or $x_2 = -1$. Substituting these values back into (i) gives respectively $x_1 = 0$ and $x_1 = -1$.

   Thus we have two fixed points, one at $(0, 0)$ and one at $(-1, -1)$.

2. **Obtain the Jacobian matrix**
   This is given by:

   $$A = \begin{pmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_1}{\partial x_2} \\[2ex] \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 2x_2 \\ 2x_1 & 1 \end{pmatrix} \tag{5.6}$$

3. **Linearisation at $(0, 0)$.**
Using Eq 5.6 we obtain the linearisation about $(0, 0)$ to be:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2x_2 \\ 2x_1 & 1 \end{pmatrix}_{(0,0)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Thus the linearisation is an **unstable star node** at the origin. The assumption will be that the nonlinear system behaves very much like this close to the origin.

4. **Linearisation at** $(-1, -1)$.
Using Eq 5.6 we obtain the linearisation about $(-1, 1)$ to be:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2x_2 \\ 2x_1 & 1 \end{pmatrix}_{(-1,-1)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

To classify this fixed point and draw its phase diagram we proceed as usual:

- The eigenvalues are given by:

$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

Thus $\lambda_1 = 3$ and $\lambda_2 = -1$. The fixed point for the linearisation is therefore a **saddle point**. Again the assumption will be that the nonlinear system is similar to the linear system in the neighbourhood of the fixed point.

- The eigenvectors are given by:
$\underline{\lambda_1 = 3}$

$$\begin{pmatrix} 1-3 & -2 \\ -2 & 1-3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

Thus we see that $-2a - 2b = 0$ which gives $\underline{E}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = -1}$

$$\begin{pmatrix} 1+1 & -2 \\ -2 & 1+1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} +2 & -2 \\ -2 & +2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

Thus we see that $+2a - 2b = 0$ which gives $\underline{E}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5. **Local phase diagram**
The phase diagrams for these two fixed points are shown in Fig 5.1. Note how the saddle is placed at $(-1, -1)$ and the directions $\underline{E_1}$ and $\underline{E_2}$ are shown relative to a set of axes (*denoted in dots*) through this point. To obtain the global phase diagram we now have to join these two diagrams together in a consistent manner. This is not always that easy and in some cases more than one construction is possible, though of course only one is correct.

The second phase diagram in Fig 5.1 is the true phase diagram for the nonlinear system, we can see how the saddle and star node obtained by linearising about the two fixed points are linked

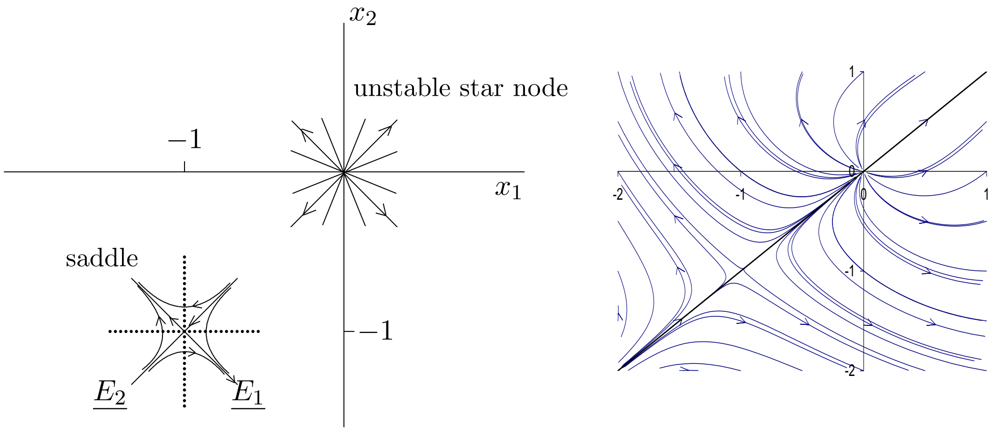


Figure 5.1: Hand drawn linear phase diagrams about the two fixed points; Nonlinear global picture

to give the global picture.

In this rather simple nonlinear problem it is possible to see that the line $x_2 = x_1$ is a trajectory in the global picture. (*shown as a heavy line in the global picture in* Fig 5.1)

To show that $x_2 = x_1$ is a trajectory we show that it is a solution of the first order differential equation defining the phase diagram as follows:

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{x_1^2 + x_2}{x_1 + x_2^2} \tag{5.7}$$

Substituting $x_2 = x_1$ into each side of this equation gives:

$$\frac{dx_2}{dx_1} = \frac{dx_1}{dx_1} = 1 \quad \text{and} \quad \frac{x_1^2 + x_2}{x_1 + x_2^2} = \frac{x_1^2 + x_1}{x_1 + x_1^2} = 1$$

Thus $x_2 = x_1$ is a solution of Eq 5.7 and hence is a trajectory in the phase plane.

Finally if we require an approximation to the solution of our nonlinear system near the fixed point $(-1, -1)$ then we can use the solution of the linearised problem about this point. In this case this is given by:

$$\underline{x} = \underline{a} + \underline{z} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + C_1 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or in component form:

$$x_1 = -1 - C_1 e^{3t} + C_2 e^{-t} \quad \text{and} \quad x_2 = -1 + C_1 e^{3t} + C_2 e^{-t}$$

To illustrate how the phase diagram might change with only a small change in the original system consider the following example:

Example 5.2

Consider the following equations:

$$\dot{x}_1 = x_1^2 + x_2 \quad \text{and} \quad \dot{x}_2 = x_1 + x_2^2$$

1. **Find the fixed points**
   The fixed points occur when:

   $$x_1^2 + x_2 = 0 \quad (i) \qquad \text{and} \qquad x_1 + x_2^2 = 0 \quad (ii)$$

   Substituting for x_2 from (i) into (ii) gives $x_1^4 + x_1 = 0$ which implies that $x_1 = 0$ or $x_1 = -1$. Substituting these values back into (i) gives respectively $x_2 = 0$ and $x_2 = -1$.

   Thus the two fixed points are still at $(0,0)$ and $(-1,-1)$.

2. **Obtain the Jacobian matrix**
   This is given by:

   $$A = \begin{pmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_1}{\partial x_2} \\ \\ \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 1 \\ \\ 1 & 2x_2 \end{pmatrix} \tag{5.8}$$

3. **Linearisation at** $(0,0)$**.**
   Using Eq 5.8 we obtain the linearisation about $(0,0)$ to be:

   $$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & 1 \\ 1 & 2x_2 \end{pmatrix}_{(0,0)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

   To classify this fixed point:

   - The eigenvalues are given by:

     $$\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = (-\lambda)^2 - 1 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0$$

     Thus $\lambda_1 = 1$ and $\lambda_2 = -1$. The fixed point for the linearisation is therefore a **saddle**.
   - The eigenvectors are given by:
     $\underline{\lambda_1 = 1}$

     $$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

     Thus we see that $-a + b = 0$ which gives $\underline{E_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

     $\underline{\lambda_2 = -1}$

     $$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

     Thus we see that $a + b = 0$ which gives $\underline{E_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

4. **Linearisation at** $(-1, -1)$.
Using Eq. (5.8) we obtain the linearisation about $(-1, 1)$ to be:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & 1 \\ 1 & 2x_2 \end{pmatrix}_{(-1,-1)} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

To classify this fixed point:

- The eigenvalues are given by:

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1) = 0$$

Thus $\lambda_1 = -3$ and $\lambda_2 = -1$. The fixed point for the linearisation is therefore a **stable node**.

- The eigenvectors are given by:
$\underline{\lambda_1 = -3}$

$$\begin{pmatrix} -2+3 & 1 \\ 1 & -2+3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

Thus we see that $a + b = 0$ which gives $\underline{E_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = -1}$

$$\begin{pmatrix} -2+1 & 1 \\ 1 & -2+1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0}$$

Thus we see that $-a + b = 0$ which gives $\underline{E_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5. **Local phase diagram**
The phase diagrams for these two fixed points are shown in Fig 5.2 Note how the stable node is placed at $(-1, -1)$ and the directions $\underline{E_1}$ and $\underline{E_2}$ are shown relative to a set of axes (*denoted in dots*) through this point. The saddle is placed at $(0, 0)$ and the directions $\underline{E_1}$ and $\underline{E_2}$ for this point are again indicated. To obtain the global phase diagram we now have to join these two diagrams together in a consistent manner.

The second phase diagram in Fig 5.2 is the true phase diagram for the nonlinear system, we can see how the saddle and stable node obtained by linearising about the two fixed points are linked to give the global picture.

As in the last problem it is possible to see that the line $x_2 = x_1$ is a trajectory in the global picture. (*shown in heavy line in the global picture in* Fig 5.2)

Finally if we require an approximation to the solution of our nonlinear system near the fixed point $(-1, -1)$ then we can again use the solution of the linearised problem about this point. In this case this is given by:

$$\underline{x} = \underline{a} + \underline{z} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + C_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

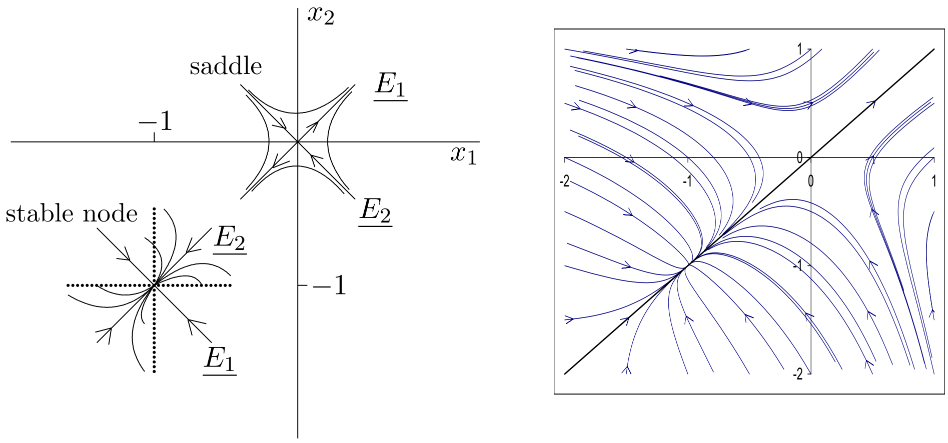


Figure 5.2: Hand drawn linear phase diagrams about the two fixed points; Nonlinear global picture

or in component form:

$$x_1 = -1 - C_1 e^{-3t} + C_2 e^{-t} \quad \text{and} \quad x_2 = -1 + C_1 e^{-3t} + C_2 e^{-t}$$

The above examples bear out what we would expect, that is the nonlinear problem and its linearisation about its fixed points have very similar phase diagrams and solutions in the neighbourhood of the fixed points. However the following example shows that this is not always the case.

Example 5.3

Given the following differential equations analyse the phase diagram and solution in the neighbourhood of the origin:

$$\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2) \qquad \dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2) \tag{5.9}$$

The equations clearly have a fixed point at the origin and thus to investigate the phase diagram near the origin we linearise the equations about $x_1 = x_2 = 0$. The Jacobian matrix is given by:

$$A = \begin{pmatrix} -3x_1^2 - x_2^2 & -1 - 2x_1x_2 \\ 1 - 2x_1x_2 & -x_1^2 - 3x_2^2 \end{pmatrix}$$

Thus at the origin the linearisation is:

$$\dot{\underline{z}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{z}$$

Evaluating the eigenvalues of A we obtain:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = +i \qquad \lambda_2 = -i$$

which indicates that the origin is a centre, or at least for the linearised problem it is. However a centre is a very delicate state for a system, it only needs a small change for the real part of the complex eigenvalue to become non-zero. In such a case the centre will turn into a focus. This example has been chosen so that we can solve it exactly and see precisely what is happening at the origin; and elsewhere for that matter. If we change to polar coordinates and substitute $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ into Eq 5.9 we obtain:

$$\dot r\cos\theta - r\dot\theta\sin\theta = -r\sin\theta - r^3\cos\theta \quad (i) \qquad \dot r\sin\theta + r\dot\theta\cos\theta = r\cos\theta - r^3\sin\theta \quad (ii)$$

Which gives, eliminating in turn $\dot r$ and $\dot\theta$ from the left hand sides of these equation:

$$(i)\times\cos\theta\ + (ii)\times\sin\theta \quad\Rightarrow\quad \dot r = -r^3 \quad\text{and}\quad (i)\times\sin\theta\ - (ii)\times\cos\theta \quad\Rightarrow\quad \dot\theta = 1$$

The equation for θ, namely, $\dot\theta = 1$, gives $\theta = t + \theta_0$, where as usual θ_0 is the value of θ at $t = 0$. This implies that as time evolves any point on a trajectory will move steadily anticlockwise about the origin. The question is does it move away from the origin, towards the origin or perform a closed path around the origin as in the case of a centre? Turning our attention to the equation for r, namely $\dot r = -r^3$ we get:

$$\dot r = -r^3 \quad\Rightarrow\quad \int -\frac{1}{r^3}\,dr = \int dt \quad\Rightarrow\quad \frac{1}{2r^2} = t + C \quad\Rightarrow\quad r = \frac{1}{\sqrt{2(t+C)}} \tag{5.10}$$

where the sign for the square root has been taken to be positive, since r, the distance from the origin is always positive or zero. From Eq 5.10 we see that as t tends to infinity r tends to zero. Thus all the trajectories will tend to and rotate anticlockwise about the the origin as t increases. Thus the origin is in truth more like a stable focus for the nonlinear problem and not a centre as indicated by linearisation. The linearised and nonlinear phase diagrams are shown in Fig 5.3. The

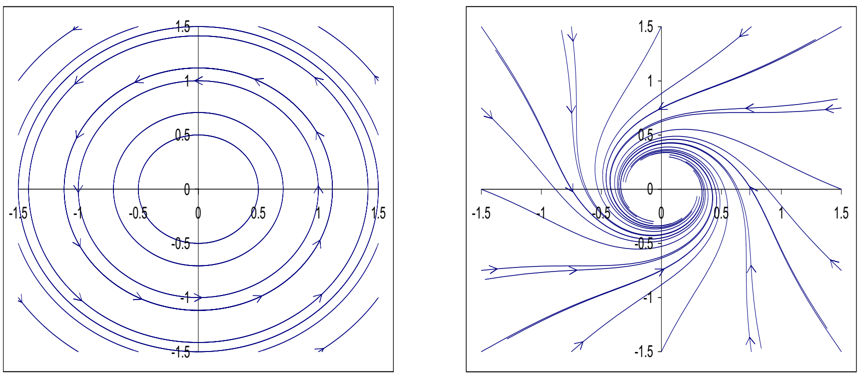


Figure 5.3: Linearised centre; Nonlinear stable focus - slow convergence to the origin

linear problem is just a canonical centre but in the anticlockwise direction whereas the original nonlinear problem looks more like a stable focus at the origin, again moving in the anticlockwise direction.

As t increases the convergence to the origin of the nonlinear trajectories is slower than that found for the stable canonical focus. The reason for this is that for the canonical focus convergence to the origin is exponential, due to the $e^{\alpha t}$ term in the solution. In this example we see from the solution of the nonlinear equation (Eq 5.10) that r clearly tends to zero as t increases, but only does so like $1/\sqrt{t}$

The above example clearly showed that a centre in the linearised problem could in reality be a stable focus in the nonlinear problem. It is also true that if the fixed point is a centre in the linearised problem it may be an unstable focus in the original nonlinear problem. Finally we must consider the question is it possible for the nonlinear and linearised versions to both be centres at a fixed point? The following example for the sine pendulum, which we saw in Chapter 1, is an example that has such fixed points, where linearising does not change the nature of the fixed points and centres in the linear problem correspond to centres in the nonlinear problem. Strictly speaking we have not defined the idea of a centre for a nonlinear system, however we will take it to mean that in every neighbourhood of the fixed point there exists closed trajectories surrounding the fixed point.

Example 5.4 - Sine pendulum

The equation of the sine pendulum considered in Chapter 1 is given by $\ddot{\theta} = -\sin\theta$ and writing this in standard form gives: $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\sin x_1$, with $x_1 = \theta$ and $x_2 = \dot{\theta}$.

The fixed points are on the x_1 axis ($x_2 = 0$) where $\sin x_1 = 0$, which implies that $x_1 = n\pi$. To linearise we form the Jacobian matrix:

$$\begin{pmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{pmatrix}$$

$$\text{Which equals} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{at} \quad x_1 = 2k\pi \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{at} \quad x_1 = (2k-1)\pi$$

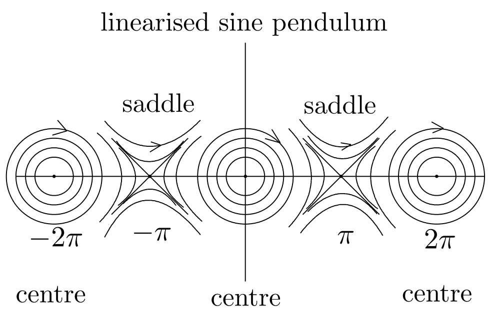
At even multiple of π (ie where $x_1 = 2k\pi$) the Jacobian is already in canonical form for a centre. At odd multiples of π it is easy to show that the Jacobian has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, thus these fixed points are saddles; further calculation gives the corresponding eigenvectors to be $\underline{E_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{E_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Using this information we are able to hand draw the first phase diagram in Fig 5.4. To obtain the phase diagram for the nonlinear problem we are able to solve the differential equation for dx_2/dx_1 as follows:

$$\frac{dx_2}{dx_1} = -\frac{\sin x_1}{x_2} \quad \Rightarrow \quad \int x_2 \, dx_2 = \int -\sin x_1 \, dx_1$$

$$\Rightarrow \quad \frac{x_2^2}{2} = \cos x_1 + C \quad \Rightarrow \quad x_2 = \pm\sqrt{2(\cos x_1 + C)} \tag{5.11}$$

This final expression generates all the trajectories in the phase plane for the nonlinear system.

Let us consider what is happening around the origin. If we consider the trajectory through the point P$(\pi/3, 1)$ then Eq 5.11 gives $C = 0$ thus $x_2 = \pm\sqrt{2\cos x_1}$. The $\pm$ in Eq 5.11 indicates that the trajectory is symmetric about the x_1-axis and since this particular curve cuts the x_1-axis at $\pm\pi/2$ the complete trajectory is a closed curve. (see Fig 5.5). As we continue to consider trajectories further and further from the origin we reach a point where $C = 1$, for example the trajectory through the point Q$(\pi/3, \sqrt{3})$. This trajectory cuts the x_1-axis at $\pm\pi$ (see Fig 5.5). Moving even further out from



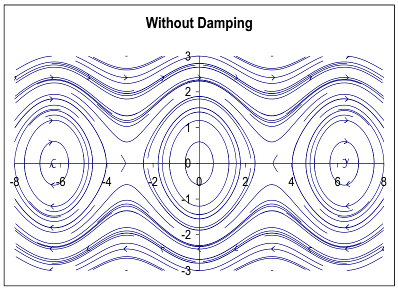


Figure 5.4: Hand drawn linearised diagram; Nonlinear sine pendulm showing the centres at even multiples of π.

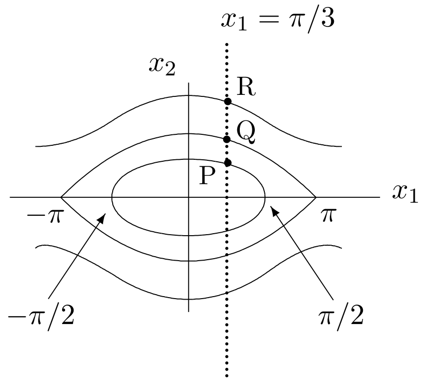


Figure 5.5:

the origin (point R) C becomes greater than 1 and therefore x_2 is never zero and the trajectories are no longer closed curves (see Fig 5.5). We have thus established that the phase diagram for the nonlinear system also has a centre at the origin. Continuing the analysis we can establish that the nonlinear system has centres at all fixed points with $x_1 = 2k\pi$, that is to say at even multiples of π. The complete phase diagram is shown in Fig 5.4, the caption non damping referring to the fact that we have ignored air resistance and friction. Below we consider the sine pendulum with damping taken into account and draw the revised phase diagram.

Example 5.5 - damped sine pendulum

This final example shows how by introducing an extra term into the equations to model the idea of friction the phase diagram changes and the centres become asymptotically stable fixed points. To model air resistance we can imagine that the faster the pendulum swings the greater the air resistance. Thus a simple assumption would be to take the resistance term to be proportional to the rate of change of theta.

Thus takeing the resistance to be equal to $\epsilon\dot{\theta}$, where ϵ is a positive constant of proportionality, the equation for the motion becomes:

$$\ddot{\theta} = -\sin\theta - \epsilon\dot{\theta}$$

which with the usual relabelling of the variables gives:

$$\dot{x}_1 = x_2 \quad \text{and} \quad \dot{x}_2 = -\sin x_1 - \epsilon x_2$$

. The fixed points are still on the x_1-axis at the same positions, that is to say at $x_1 = n\pi$ To linearise we form the Jacobian matrix:

$$\begin{pmatrix} 0 & 1 \\ -\cos x_1 & -\epsilon \end{pmatrix} \quad \text{which} \quad = \begin{pmatrix} 0 & 1 \\ -1 & -\epsilon \end{pmatrix} \quad x_1 = 2k\pi \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & -\epsilon \end{pmatrix} \quad x_1 = (2k-1)\pi$$

$\underline{x_1 = (2k-1)\pi}$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\epsilon - \lambda \end{vmatrix} = \lambda^2 + \epsilon\lambda - 1 = 0 \quad \Rightarrow \quad \lambda = \frac{-\epsilon \pm \sqrt{\epsilon^2 + 4}}{2}$$

For all ϵ this give two non-zero values of λ each of different sign. Thus we still have a saddle point at $x_1 =$ odd multiples of π.

$\underline{x_1 = 2k\pi}$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\epsilon - \lambda \end{vmatrix} = \lambda^2 + \epsilon\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{-\epsilon \pm \sqrt{\epsilon^2 - 4}}{2}$$

Depending on the value of ϵ we obtain the following types of fixed point at $x_1 =$ even multiples of π.

- $\epsilon^2 - 4 < 0$: this gives two complex eigenvalues with real part $= -\epsilon/2$, thus since $\epsilon > 0$ the fixed point is a stable focus. The phase diagram for this case is given in Fig 5.6 where ϵ has been taken equal to 1.
- $\epsilon^2 - 4 = 0$: this gives a single eigenvalue $= -\epsilon/2$ and hence the fixed point is an stable improper node since $\epsilon > 0$.
- $\epsilon^2 - 4 > 0$: with $\epsilon > 0$ this give two real negative eigenvalues, thus the fixed point is a stable node.

We note that no matter what positive value ϵ takes the fixed points at the even multiples of π are stable and the pendulum will ultimately tend to rest at one of these points.

5.2 Linearisation theorem

In section 5.1 we saw how the linearisation of a nonlinear system about its fixed points could be used to construct the global phase diagram for the system. We also saw that not in all cases did the linearised system and the nonlinear system have the same phase diagram around a fixed point. This section sets out in the form of definitions and theorems what we can and cannot do as far as linearisation is concerned.

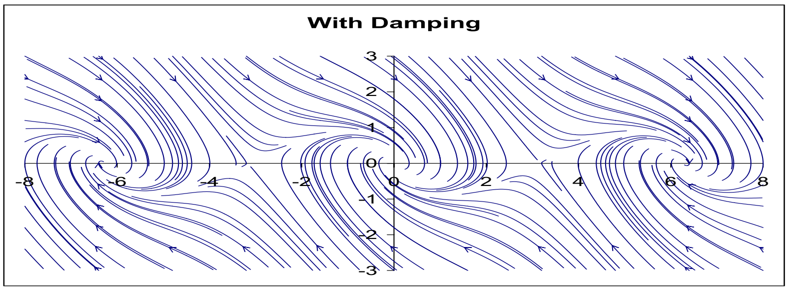


Figure 5.6: Damped sine pendulum: $\ddot{\theta} = -\sin\theta - \dot{\theta}$; stable focii at even multiples of π saddles at odd multiples of π

Definition - simple fixed point
The system

$$\underline{\dot{x}} = \underline{X}(\underline{x})$$

is said to have a **simple fixed point** at $\underline{x} = \underline{a}$ if:

- $\underline{X}(\underline{a}) = 0$, that is to say $\underline{x} = \underline{a}$ is a fixed point.
- The linearisation of the system about $\underline{x} = \underline{a}$ is simple. That is to say if the linearisation is $\underline{\dot{z}} = A\underline{z}$ then $\det A \neq 0$.

The idea of an autonomous linear system having a simple fixed point has now been extended via the concept of linearisation to the idea of any autonomous system having a simple fixed point. We now state the main linearisation theorem:

Theorem - Linearisation
In the neighbourhood of a simple fixed point of the nonlinear system

$$\underline{\dot{x}} = \underline{X}(\underline{x})$$

the phase portrait of the system and its linearisation are similar, except possibly in the case where the linearised system is a centre.

Classification of fixed points
We use the classification of the linearised system to classify a simple fixed point of a nonlinear system, except where the linearisation is a centre.

As we have seen if the linearisation is a centre at the fixed point the nonlinear system may also be a centre in the sense that it is surrounded by closed trajectories. Alternatively the nonlinear system may also be a stable or unstable focus in the sense that the trajectories about the fixed point spiral into or out of the fixed point (next year, if you do the 2nd part of this module, you will learn more about this special cases known as *limit cycles*).