Entanglement Entropy in Massive Quantum Field Theories

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May 2015

Overview of the talk

- Introduction and Definition of Entanglement Entropy

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- Entanglement Entropy from Twist Fields

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- Conclusions and Outlook

Our contribution so far...

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D. Bianchini, OC-A, B. Doyon, arXiv:1502.03275
D. Bianchini, OC-A, B. Doyon, E. Levi and F. Ravanini, J. Phys. A48 04FT01 (2015) (IoP Select)
E. Levi, OC-A and B. Doyon, Phys. Rev. B 88 094439 (2013)
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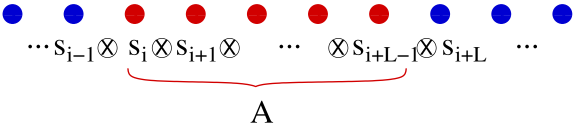

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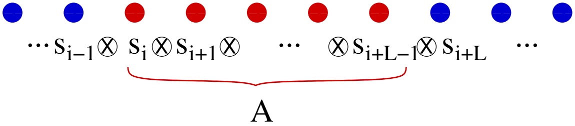

then we define

Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

$|\Psi\rangle$ ground state and ρ_A the reduced density matrix.

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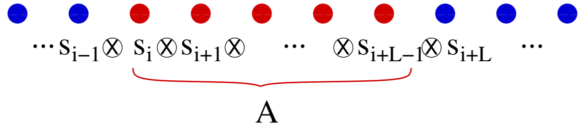

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Other Entropies

$$S_A^{\text{Rényi}} = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}, \qquad S_A^{\text{Tsallis}} = \frac{1-\mathrm{Tr}_A(\rho_A^n)}{n-1}$$

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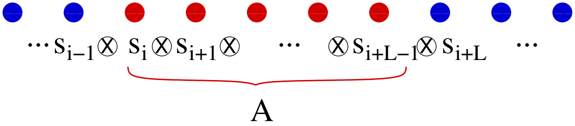

Replica Trick

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_A(\rho_A^n)$$

- For general QFTs the "replica trick" naturally leads to the notion of replica theories on multi-sheeted Riemann surfaces $\Rightarrow$ interpretation of $\mathrm{Tr}_A(\rho_A^n)$

Motivation

The study of the EE of extended quantum systems is a popular area of research in various areas:

- **Quantum Information:** The EE quantifies the amount of “surprise” that a sub-part of a system finds when discovering it is correlated to the rest of the system. Therefore, entanglement entropy is a *bona fide* measure of the correlations in the system [Latorre & Riera, Review'09]

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- **Quantum Field Theory:** The EE can be defined for any QFT (operator independent). It provides "universal" information about quantum systems/quantum states. [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Latorre, Rico & Kitaev'03; Latorre, Rico & Vidal'04; Calabrese & Cardy '04; J.L. Cardy, O.C-A & B. Doyon'08]

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- **Holography (AdS/CFT Correspondence):** The EE in the CFT is given by the area of a certain extremal surface in the bulk (AdS) [Ryu and Takayanagi'06]

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- Near critical points (e.g. Conformal Field Theory) it displays universal behaviour.
- In the context of high energy physics much of the motivation to study the entanglement entropy has come from its behaviour at quantum critical points [Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04]:

$$S(L) \sim \frac{c}{3} \log L \quad \Rightarrow \quad \text{information about the CFT}$$

e.g. the EE of a subsystem of length L diverges logarithmically. The proportionality constant c is called the central change. It uniquely characterises the CFT.

Example: the Ising model

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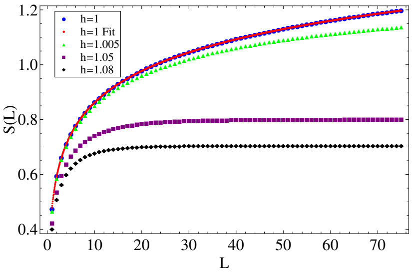

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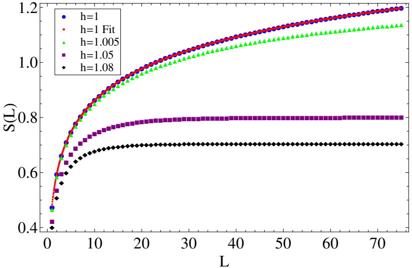
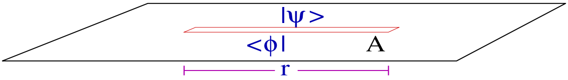


- $S(L) = \frac{0.500003}{3}\log L + 0.478551$ for $h = 1$. For $h > 1$ saturation is reached [Vidal, Latorre, Rico & Kitaev'03].

Partition functions on multi-sheeted Riemann surfaces

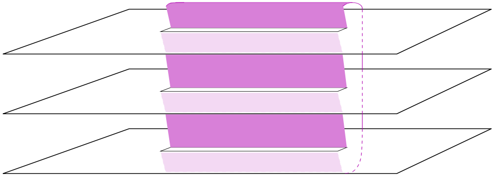
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04] ($\mathrm{Tr}_A(\rho_A)$ is the partition function of the original theory!):

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n =$$



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- The first approach does not extend beyond critical systems, as it uses conformal maps/symmetry
- Also, even in CFT mapping to the complex plane is only possible for one single interval
- However *it is possible to define correlation functions in any Quantum Field Theory* (QFT) and so expressing the EE in terms of correlation functions of twist fields provides a method which may be extended beyond CFT

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- We call these theories integrable models (e.g. sine-Gordon, Lee-Yang theories) and they come with a set of "tools" for computing correlation functions which makes them particularly attractive
- Some of our main results do also hold for generic 1+1 dimensional QFTs [B. Doyon'09]

Short and Long Distance Behaviour

- Recall that [Cardy, OC-A & Doyon'08]:

$$Z_n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn}\frac{Z_n}{Z_1^n}$$

where D_n is a normalisation constant ($D_1 = Z_1$ & $D_1' = 0$), and d_n is the conformal scaling dimension of $\mathcal{T}$ [Knizhnik'87; Calabrese & Cardy'04]:

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- Large distance: $0 \ll \xi \ll r$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \sim \langle \mathcal{T}\rangle_n^2 = g_n^2 m^{2d_n} \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m\varepsilon) + U$$

with $U = -2g_1'$.

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$$\begin{aligned} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \text{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\text{gs}\rangle \\ &= \sum_{\text{state } k} \langle \text{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\text{gs}\rangle \end{aligned}$$

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- The matrix elements $\langle \text{gs}|\mathcal{T}(0)|k\rangle$ are called *form factors*
- For integrable models, an specific program exists (*form factor program*) that allows their exact computation
- However the program needs to be modified to include twist fields correctly

Particles and States in 1+1 dimensional QFT

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- These *in*- or *out*-states are denoted by $|\theta_1, \theta_2, \ldots, \theta_k\rangle^{in,out}_{\mu_1,\mu_2,\ldots,\mu_k}$ with $\theta_1 > \ldots > \theta_k$ for *in*-states and the opposite for *out*-states, where θ_i's are rapidities and μ_i's are particle types

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- Energy and momentum of these states are the sums of those of individual particles: $E = \sum_{i=0}^{k} m_{\mu_i} \cosh\theta_i$ and $P = \sum_{i=0}^{k} m_{\mu_i} \sinh\theta_i$.
- In terms of these states, the generic state $|k\rangle$ in our form factor (FF) expansion is:

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- The quantum numbers $\mu_1, \ldots, \mu_k$ will label the copy number in the replica theory

Form Factor Programme for Twist Fields

- Recall the general commutation relations with the fundamental fields of the replica model:

$$
\begin{aligned}
\varphi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\varphi_{i+1}(y) \qquad x^1 > y^1, \\
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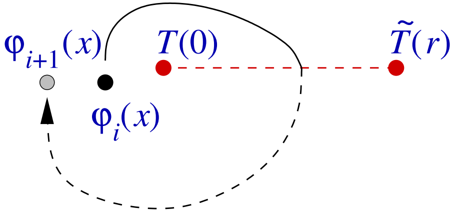
for $i = 1, \ldots, n$ and $n + i \equiv i$.

- Diagramatically:



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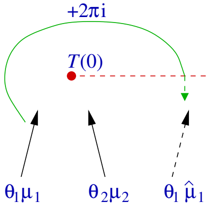

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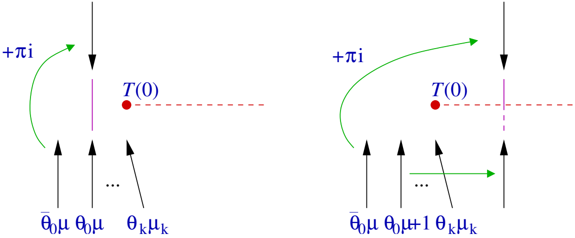

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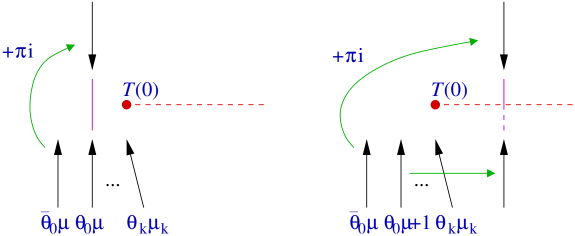

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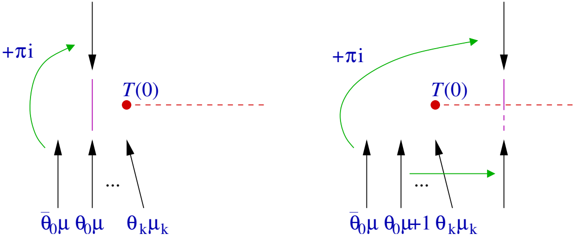

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- These equations can be solved recursively as they relate lower- to higher-particle form factors

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$$\begin{aligned}
\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\
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- Here we are considering a theory with vanishing one-particle form factor (even if it was non-vanishing it would not change the result for the entropy)
- Main difficulty: analytically continue $f(\theta,n)$ for $n \in \mathbb{R}$, $n \leq 1$, then take the derivative at $n = 1$.

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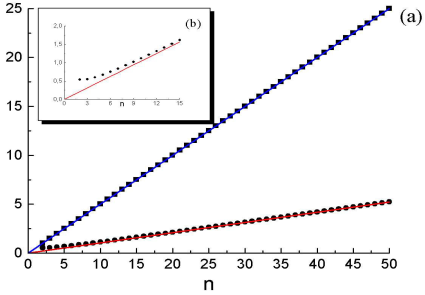

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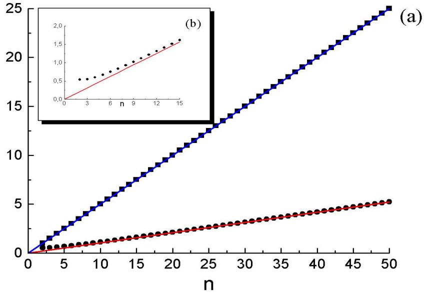

- Problem: the analytic continuation $\tilde{f}(\theta, n)$ of $f(\theta, n)$ does not converge uniformly as $n \to 1$ on $\theta \in \mathbb{R}$, that is, $\tilde{f}(0, 1) \neq f(0, 1) = 0$

Analytic Continuation Resolved

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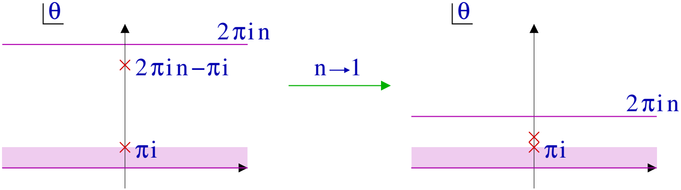

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- Extracting the poles and resuming in j exactly gives:

$$\tilde{f}(\theta,n) \sim_{n\to 1} \tilde{f}(0,1)\left(\frac{i\pi(n-1)}{2(\theta+i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta-i\pi(n-1))}\right)$$

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with

$$\tilde{f}(0,1) = \frac{1}{2}$$

- Hence the derivative is supported at $\theta = 0$:

$$\frac{d}{dn}\left(n\tilde{f}(\theta,n)\right)_{n=1} = \pi^2 \tilde{f}(0,1)\delta(\theta)$$

- This gives the universal correction to saturation:

$$-\lim_{n\to 1}\frac{d}{dn}\left(\frac{n}{4\pi^2}\int_{-\infty}^{\infty} f(\theta,n)K_0(2mr\cosh(\theta/2)d\theta + \ldots\right) = -\frac{1}{8}K_0(2mr)$$

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- There is however no general understanding on how to perform the analytic continuation for higher particle form factors in interacting theories (we completely understood this for the Ising model in [O.C-A & B. Doyon'09])
- The problem is also not fully solved for CFT for more complicated geometries (e.g. several disconnected regions [P. Calabrese, J.L. Cardy & E. Tonni'09])

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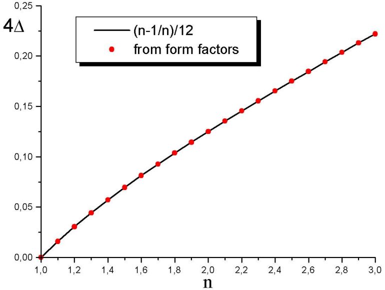

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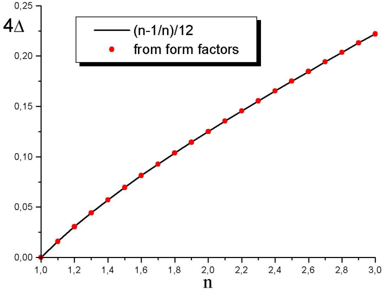

- Here we are examining the short-distance behaviour of the two-point function of twist fields from a FF expansion for the Ising model [O.C-A & B. Doyon'09]

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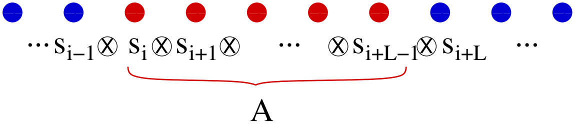
$$S(r) = -\frac{c_{\text{eff}}}{3}\log(m\epsilon) + U - \underbrace{\frac{2}{\pi f(\frac{2\pi i}{3},1)^2}\left(\frac{1}{\sqrt{3}} - \frac{13\pi}{108}\right)}_{0.0769782} K_0(mr) + \cdots$$

- Let us go back to our original set-up, that is the bi-partitioned quantum spin chain



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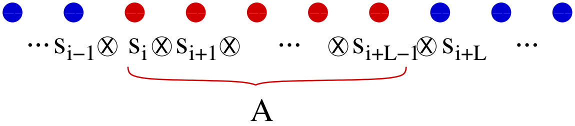

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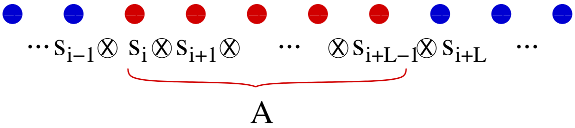

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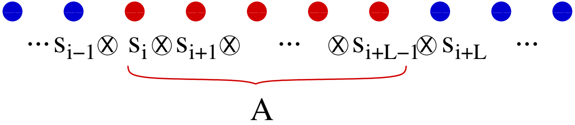

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- Using either exact diagonalization (for Ising) or DMRG (for XXZ) we may compute the EE for each of these models near (but away from) their critical points [O.C-A, B. Doyon & E. Levi'12]

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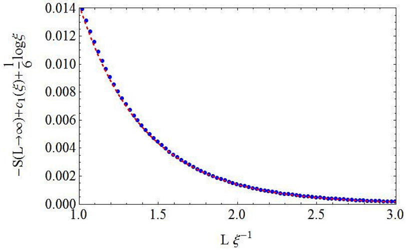

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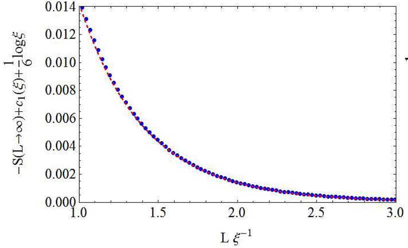

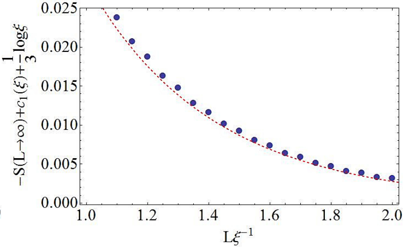

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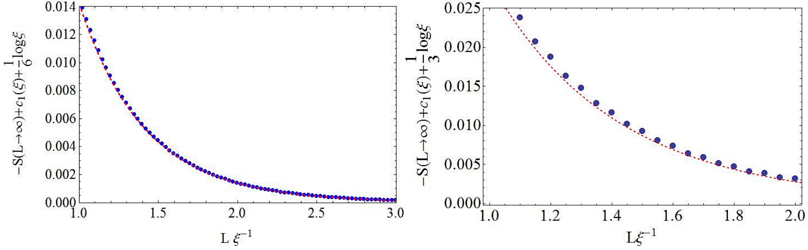

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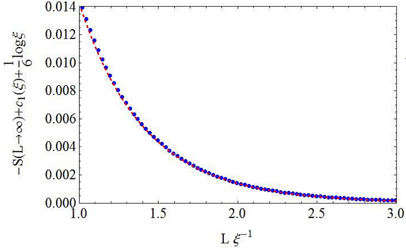

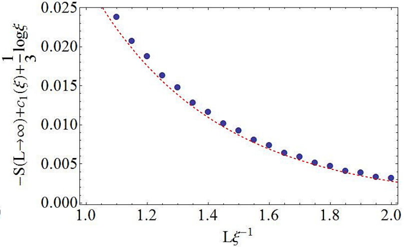

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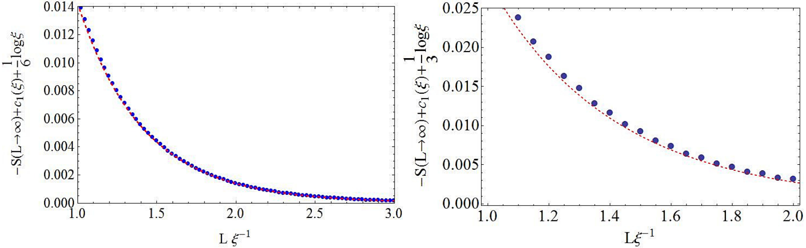

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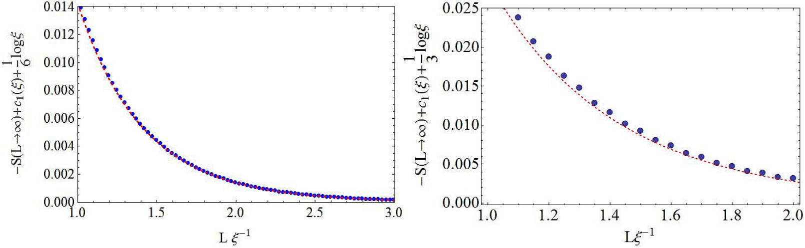

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- In the second figure, numerical results are fitted to the function $1/4K_0(2L/\xi)$ with good agreement. This is what we would expect for the sine-Gordon model, which has two fundamental particles of equal mass

Conclusions

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- Many open problems remain in this area which is still dominated by the investigation of critical systems
- A natural next step is to look at other measures of entanglement which are more natural for mixed states (e.g. negativity), consider the EE of excited states and/or disconnected regions