CITY UNIVERSITY
LONDON
EST 1894

Universal Features of the Negativity of 1+1 dimensional Quantum Field Theories

Olalla A. Castro-Alvaredo

School of Mathematics, Computer Science and Engineering
Department of Mathematics
City University London

School of Mathematics, University of Leeds
Integrable Systems Seminar
October 2015

Background:

Background:

- This talk will mainly be a review of the most important research directions regarding measures of entanglement in 1+1 dimensional relativistic QFTs.

Background:

- This talk will mainly be a review of the most important research directions regarding measures of entanglement in 1+1 dimensional relativistic QFTs.
- My main contribution to the topic of Logarithmic Negativity is:

  Olivier Blondeau-Fournier, OCA and Benjamin Doyon, *Universal scaling of the logarithmic negativity in massive quantum field theory*, arXiv:1508.04026.

Background:

- This talk will mainly be a review of the most important research directions regarding measures of entanglement in 1+1 dimensional relativistic QFTs.
- My main contribution to the topic of Logarithmic Negativity is:

  Olivier Blondeau-Fournier, OCA and Benjamin Doyon, *Universal scaling of the logarithmic negativity in massive quantum field theory*, arXiv:1508.04026.

- Throughout the talk I will also refer to some previous work, especially our first paper on the subject:

  John L. Cardy, OCA and Benjamin Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, J. Stat. Phys. 130 (2008) 129-168.

Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.

Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.
  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\ ,\quad \langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.
  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \;, \quad \langle \hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by **local variables**.
- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

$$|\hat{\psi}\rangle = \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

$$= \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right)$$

- States of this type are known as **pure states**.

- States of this type are known as **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ , \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

(for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).

- States of this type are known as **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ , \quad \langle \hat{A} \rangle = \mathrm{Tr}(\rho \hat{A})$$

  (for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).

- These examples are extremely simple but what happens in extended many-body quantum systems?

- States of this type are known as **pure states**. Mixed states are described by density matrices

$$\rho = \sum_{\alpha} p_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}| \;, \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

  (for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).
- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]

- States of this type are known as **pure states**. Mixed states are described by density matrices

$$\rho = \sum_{\alpha} p_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}| \ , \quad \langle \hat{A} \rangle = \mathrm{Tr}(\rho \hat{A})$$

  (for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).
- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
  1. Should be an entanglement monotone: should not increase under LOCC

- States of this type are known as **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ , \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

(for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).
- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
  1. Should be an entanglement monotone: should not increase under LOCC
  2. Should be invariant under unitary transformations

- States of this type are known as **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ , \quad \langle \hat{A} \rangle = \mathrm{Tr}(\rho \hat{A})$$

(for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).
- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
    1. Should be an entanglement monotone: should not increase under LOCC
    2. Should be invariant under unitary transformations
    3. Should vanish for separable states

- States of this type are known as **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \;, \quad \langle \hat{A} \rangle = \mathrm{Tr}(\rho \hat{A})$$

(for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).
- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
    1. Should be an entanglement monotone: should not increase under LOCC
    2. Should be invariant under unitary transformations
    3. Should vanish for separable states
    4. Should not vanish for non-separable states

- States of this type are known as **pure states**. Mixed states are described by density matrices

$$\rho = \sum_{\alpha} p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ , \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

(for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).

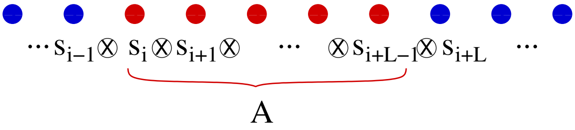
- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
- The bi-partite entanglement entropy [Bennett et al.'96] and the logarithmic negativity [Vidal & Werner'01; Plenio'05] are good measures of entanglement according to these properties

Bi-partite Entanglement Entropy (EE)

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$

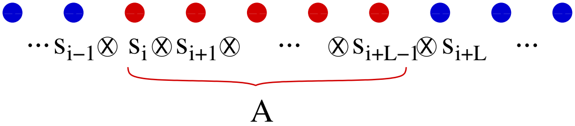
Bi-partite Entanglement Entropy (EE)

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$



Bi-partite Entanglement Entropy (EE)

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$



then we define

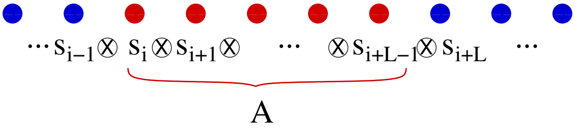
Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

$|\Psi\rangle$ ground state and ρ_A the reduced density matrix.

Bi-partite Entanglement Entropy (EE)

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$



then we define

Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

$|\Psi\rangle$ ground state and ρ_A the reduced density matrix.

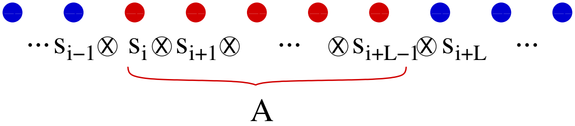
- Other entropies may also be defined such as

Other Entropies

$$S_A^{\text{Rényi}} = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}, \qquad S_A^{\text{Tsallis}} = \frac{1-\mathrm{Tr}_A(\rho_A^n)}{n-1}$$

Bi-partite Entanglement Entropy (EE)

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N-L$



Replica Trick

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$$

- For general QFTs the “replica trick” naturally leads to the notion of replica theories on multi-sheeted Riemann surfaces $\Rightarrow$ interpretation of $\mathrm{Tr}_A(\rho_A^n)$

Some Motivation to Study the EE in QFT

Some Motivation to Study the EE in QFT

- The EE provides information about the state of a quantum system

Some Motivation to Study the EE in QFT

- The EE provides information about the state of a quantum system
- At critical points (CFT) and near critical points (QFT) it displays universal behaviour

Some Motivation to Study the EE in QFT

- The EE provides information about the state of a quantum system
- At critical points (CFT) and near critical points (QFT) it displays universal behaviour
- The best known motivation to study the EE relates to its behaviour at quantum critical points [Holzhey, Larsen & Wilczek'94; Vidal, Latorre, Rico & Kitaev'03; Calabrese & Cardy'04; Bianchini et al.'15]:

$$S(L) \sim \frac{c_{\text{eff}}}{3} \log L \quad \Rightarrow \quad \text{information about the CFT}$$

c_{eff} is the (effective) central change which uniquely characterises the CFT

Some Motivation to Study the EE in QFT

- The EE provides information about the state of a quantum system
- At critical points (CFT) and near critical points (QFT) it displays universal behaviour
- The best known motivation to study the EE relates to its behaviour at quantum critical points [Holzhey, Larsen & Wilczek'94; Vidal, Latorre, Rico & Kitaev'03; Calabrese & Cardy'04; Bianchini et al.'15]:

$$S(L) \sim \frac{c_{\text{eff}}}{3} \log L \quad \Rightarrow \quad \text{information about the CFT}$$

  c_{eff} is the (effective) central change which uniquely characterises the CFT
- Computing the EE is claimed to be the most efficient numerical approach to classifying critical points!

Example: the Ising model

$$H = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z\right)$$

Example: the Ising model

$$H = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + h\sigma_i^z\right)$$

- We may carry out the "scaling limit" of this theory in two different ways:

Example: the Ising model

$$H = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + h\sigma_i^z\right)$$

- We may carry out the "scaling limit" of this theory in two different ways:
- Set $h = 1$ from the beginning: then $\xi = \infty$ and in the limit $N \to \infty$ this is a critical model.

Example: the Ising model

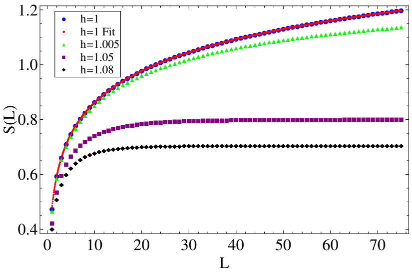
$$H = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + h\sigma_i^z\right)$$

- We may carry out the "scaling limit" of this theory in two different ways:
- Set $h=1$ from the beginning: then $\xi=\infty$ and in the limit $N\to\infty$ this is a critical model.
- Take $h>1$: $\xi\propto m^{-1}$ finite but large. Taking $N\to\infty$ while L/ξ is finite we obtain *Ising field theory*.

Example: the Ising model

$$H = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z\right)$$

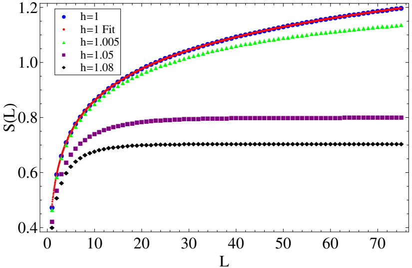
- We may carry out the "scaling limit" of this theory in two different ways:
- Set $h = 1$ from the beginning: then $\xi = \infty$ and in the limit $N \to \infty$ this is a critical model.
- Take $h > 1$: $\xi \propto m^{-1}$ finite but large. Taking $N \to \infty$ while L/ξ is finite we obtain *Ising field theory.*



Example: the Ising model

$$H = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z\right)$$

- We may carry out the "scaling limit" of this theory in two different ways:
- Set $h = 1$ from the beginning: then $\xi = \infty$ and in the limit $N \to \infty$ this is a critical model.
- Take $h > 1$: $\xi \propto m^{-1}$ finite but large. Taking $N \to \infty$ while L/ξ is finite we obtain *Ising field theory*.



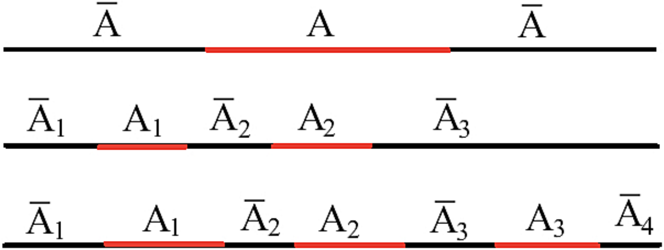
- $S(L) = \frac{0.500003}{3}\log L + 0.478551$ for $h = 1$. For $h > 1$ saturation is reached [Vidal, Latorre, Rico & Kitaev'03; Levi, OCA, Doyon'12].

More complex configurations

- Everything we have said so far refers to the EE of *one interval*. If the regions A and $\bar{A}$ are not simply connected, then the EE is much more difficult to compute.

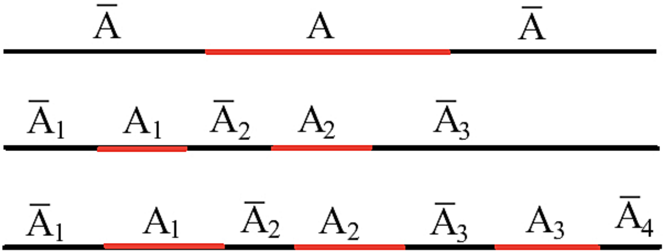
More complex configurations

- Everything we have said so far refers to the EE of *one interval*. If the regions A and $\bar{A}$ are not simply connected, then the EE is much more difficult to compute.



More complex configurations

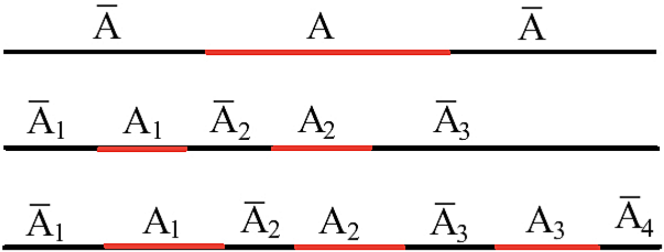
- Everything we have said so far refers to the EE of *one interval*. If the regions A and $\bar{A}$ are not simply connected, then the EE is much more difficult to compute.



- The figures represent the one interval, double interval and triple interval configurations.

More complex configurations

- Everything we have said so far refers to the EE of *one interval*. If the regions A and $\bar{A}$ are not simply connected, then the EE is much more difficult to compute.



- The figures represent the one interval, double interval and triple interval configurations.
- These configurations have been studied by various people in CFT, especially in the works of [Calabrese, Cardy and Tonni'12'13'14].

Logarithmic Negativity (LN)

Logarithmic Negativity (LN)

- The EE is a good measure of entanglement for pure states. What about mixed states?

Logarithmic Negativity (LN)

- The EE is a good measure of entanglement for pure states. What about mixed states?
- The LN provides a good measure of entanglement in mixed states for non-complementary regions such as A and B [Vidal, Werner'01; Plenio'05]

Logarithmic Negativity (LN)

- The EE is a good measure of entanglement for pure states. What about mixed states?
- The LN provides a good measure of entanglement in mixed states for non-complementary regions such as A and B [Vidal, Werner'01; Plenio'05]

Logarithmic Negativity (LN)

- The EE is a good measure of entanglement for pure states. What about mixed states?
- The LN provides a good measure of entanglement in mixed states for non-complementary regions such as A and B [Vidal, Werner'01; Plenio'05]

Logarithmic Negativity

$$\mathcal{E} = \log \mathrm{Tr}_{A\cup B}|\rho_{A\cup B}^{T_B}| \quad \text{with} \quad \rho_{A\cup B} = \mathrm{Tr}_C(|\Psi\rangle\langle\Psi|)$$

- Where $\mathrm{Tr}|\rho|$ represents the sum of the absolute values of the eigenvalues of ρ and T_B represents "partial transposition"

Logarithmic Negativity (LN)

- The EE is a good measure of entanglement for pure states. What about mixed states?
- The LN provides a good measure of entanglement in mixed states for non-complementary regions such as A and B [Vidal, Werner'01; Plenio'05]

Logarithmic Negativity

$$\mathcal{E} = \log \mathrm{Tr}_{A\cup B}|\rho_{A\cup B}^{T_B}| \quad \text{with} \quad \rho_{A\cup B} = \mathrm{Tr}_C(|\Psi\rangle\langle\Psi|)$$

- Where $\mathrm{Tr}|\rho|$ represents the sum of the absolute values of the eigenvalues of ρ and T_B represents "partial transposition"
- $|\Psi\rangle$ is the state of the whole system (for pure states)

Logarithmic Negativity (LN)

- The EE is a good measure of entanglement for pure states. What about mixed states?
- The LN provides a good measure of entanglement in mixed states for non-complementary regions such as A and B [Vidal, Werner'01; Plenio'05]

- There is also a "replica" approach to the computation of the negativity [Calabrese, Cardy & Tonni'12]:

Logarithmic Negativity from the Replica Trick

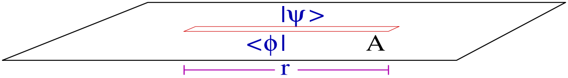
$$\mathcal{E}[n] = \log \mathrm{Tr}_{A\cup B}(\rho_{A\cup B}^{T_B})^n \quad \text{then} \quad \mathcal{E} = \lim_{n\to 1} \mathcal{E}_e[n]$$

where $\mathcal{E}_e[n]$ means the function $\mathcal{E}[n]$ for n even. This limit requires analytic continuation from n even to $n=1$

Partition functions on multi-sheeted Riemann surfaces

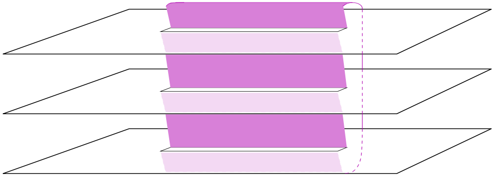
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04] ($\mathrm{Tr}_A(\rho_A)$ is the partition function of the original theory!):

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\; \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n =$$



Branch Point Twist Fields

- For general 1+1 dimensional QFT we have found [Calabrese, Cardy'04; Cardy, OCA & Doyon'08] that the EE may be expressed in terms of a two-point function of twist fields:

$$Z_n = D_n \varepsilon^{4\Delta_n} \langle \mathcal{T}(0) \tilde{\mathcal{T}}(r) \rangle_n \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

Branch Point Twist Fields

- For general 1+1 dimensional QFT we have found [Calabrese, Cardy'04; Cardy, OCA & Doyon'08] that the EE may be expressed in terms of a two-point function of twist fields:

$$Z_n = D_n \varepsilon^{4\Delta_n} \langle \mathcal{T}(0) \tilde{\mathcal{T}}(r) \rangle_n \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

  where D_n is a normalisation constant, and Δ_n is the conformal dimension of $\mathcal{T}$ [Knizhnik'87; Dixon et al.'87; Calabrese & Cardy'04]:

$$\Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

Branch Point Twist Fields

- For general 1+1 dimensional QFT we have found [Calabrese, Cardy'04; Cardy, OCA & Doyon'08] that the EE may be expressed in terms of a two-point function of twist fields:

$$Z_n = D_n \varepsilon^{4\Delta_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

  where D_n is a normalisation constant, and Δ_n is the conformal dimension of $\mathcal{T}$ [Knizhnik'87; Dixon et al.'87; Calabrese & Cardy'04]:

$$\Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll r \ll \xi$, logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \sim r^{-4\Delta_n} \ \Rightarrow \ S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

Branch Point Twist Fields

- For general 1+1 dimensional QFT we have found [Calabrese, Cardy'04; Cardy, OCA & Doyon'08] that the EE may be expressed in terms of a two-point function of twist fields:

$$Z_n = D_n \varepsilon^{4\Delta_n} \langle \mathcal{T}(0) \tilde{\mathcal{T}}(r) \rangle_n \,, \quad S_A = -\lim_{n \to 1} \frac{d}{dn} Z_n$$

where D_n is a normalisation constant, and Δ_n is the conformal dimension of $\mathcal{T}$ [Knizhnik'87; Dixon et al.'87; Calabrese & Cardy'04]:

$$\Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll r \ll \xi$, logarithmic behavior

$$\langle \mathcal{T}(0) \tilde{\mathcal{T}}(r) \rangle_n \sim r^{-4\Delta_n} \;\Rightarrow\; S_A \sim \frac{c}{3} \log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll r$, saturation

$$\langle \mathcal{T}(0) \tilde{\mathcal{T}}(r) \rangle_n \sim \langle \mathcal{T} \rangle_n^2 \;\Rightarrow\; S_A \sim -\frac{c}{3} \log(m\varepsilon) + U$$

Main Properties of Twist Fields

- The Twist Fields are defined through very general commutation relations with the fundamental field of the model [Cardy, OCA & Doyon'08]:

$$\begin{aligned}\Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_{i+1}(y) \qquad x^1 > y^1,\\ \Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_i(y) \qquad x^1 < y^1,\end{aligned}$$

$$\begin{aligned}\Phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Phi_{i-1}(y) \qquad x^1 > y^1,\\ \Phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Phi_i(y) \qquad x^1 < y^1.\end{aligned}$$

for $i = 1, \ldots, n$ and $n + i \equiv i$.

Main Properties of Twist Fields

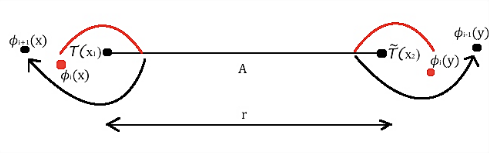
- The Twist Fields are defined through very general commutation relations with the fundamental field of the model [Cardy, OCA & Doyon'08]:

$$\begin{aligned}
\Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_{i+1}(y) \qquad x^1 > y^1, \\
\Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_i(y) \qquad x^1 < y^1,
\end{aligned}$$

$$\begin{aligned}
\Phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Phi_{i-1}(y) \qquad x^1 > y^1, \\
\Phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Phi_i(y) \qquad x^1 < y^1.
\end{aligned}$$

for $i = 1, \ldots, n$ and $n + i \equiv i$.

- Diagramatically:



Entropy from Form Factors

- The two-point function of branch-point twist fields can be decomposed into the *in*-basis, giving a large-distance expansion:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n = \langle \mathrm{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{vac}\rangle =$$

$$\sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k=1}^{n}\int \frac{d\theta_1\cdots d\theta_k}{(2\pi)^k}\,|F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)|^2 e^{-r\sum\limits_{i=1}^{k} m_{\mu_i}\cosh\theta_i}$$

where

$$F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k) = \langle \mathrm{vac}|\mathcal{T}(0)|\theta_1,\ldots,\theta_k\rangle^{in}_{\mu_1,\ldots,\mu_k}$$

are the k-particle form factors of the twist-field $\mathcal{T}$.

Entropy from Form Factors

- The two-point function of branch-point twist fields can be decomposed into the *in*-basis, giving a large-distance expansion:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n = \langle \mathrm{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{vac}\rangle =$$

$$\sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k=1}^{n}\int \frac{d\theta_1\cdots d\theta_k}{(2\pi)^k}\,|F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)|^2 e^{-r\sum\limits_{i=1}^{k} m_{\mu_i}\cosh\theta_i}$$

where

$$F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k) = \langle \mathrm{vac}|\mathcal{T}(0)|\theta_1,\ldots,\theta_k\rangle^{in}_{\mu_1,\ldots,\mu_k}$$

are the k-particle form factors of the twist-field $\mathcal{T}$.

- Typically the expansion is rapidly convergent in k for large r (short-distance expansion).

Entropy from Form Factors

- The two-point function of branch-point twist fields can be decomposed into the *in*-basis, giving a large-distance expansion:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n = \langle \text{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\text{vac}\rangle =$$

$$\sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k=1}^{n}\int \frac{d\theta_1\cdots d\theta_k}{(2\pi)^k}\,|F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)|^2 e^{-r\sum\limits_{i=1}^{k} m_{\mu_i}\cosh\theta_i}$$

where

$$F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k) = \langle \text{vac}|\mathcal{T}(0)|\theta_1,\ldots,\theta_k\rangle^{in}_{\mu_1,\ldots,\mu_k}$$

are the k-particle form factors of the twist-field $\mathcal{T}$.
- Typically the expansion is rapidly convergent in k for large r (short-distance expansion).
- These form factors can be computed as the solutions to a set of consistency equations which we formulated in our first paper [Cardy, OCA & Doyon'08].

Logarithmic Negativity from Twist Fields

- The twist field approach has been used in the study of the LN of CFT [Calabrese, Cardy & Tonni'12'13'14]

Logarithmic Negativity from Twist Fields

- The twist field approach has been used in the study of the LN of CFT [Calabrese, Cardy & Tonni'12'13'14]

Logarithmic Negativity from Twist Fields

$$\mathcal{E}[n] = \log\left(\varepsilon^{8\Delta_n}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\mathcal{T}(r_4)\rangle_n\right)$$

Logarithmic Negativity from Twist Fields

- The twist field approach has been used in the study of the LN of CFT [Calabrese, Cardy & Tonni'12'13'14]

Logarithmic Negativity from Twist Fields

$$\mathcal{E}[n] = \log\left(\varepsilon^{8\Delta_n}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\mathcal{T}(r_4)\rangle_n\right)$$

- This 4-point function has been investigated in CFT but the analytic continuation remains challenging, even for free theories.

Logarithmic Negativity from Twist Fields

- The twist field approach has been used in the study of the LN of CFT [Calabrese, Cardy & Tonni'12'13'14]

Logarithmic Negativity from Twist Fields

$$\mathcal{E}[n] = \log\left(\varepsilon^{8\Delta_n}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\mathcal{T}(r_4)\rangle_n\right)$$

- This 4-point function has been investigated in CFT but the analytic continuation remains challenging, even for free theories.
- An interesting limit is $\lim_{r_2\to r_3}\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\sim\tilde{\mathcal{T}}^2(r_3)$ where $\tilde{\mathcal{T}}^2$ is defined as the twist field associated to the cyclic permutation $j\mapsto j-2$.

Logarithmic Negativity from Twist Fields

- The twist field approach has been used in the study of the LN of CFT [Calabrese, Cardy & Tonni'12'13'14]

Logarithmic Negativity from Twist Fields

$$\mathcal{E}[n] = \log\left(\varepsilon^{8\Delta_n}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\mathcal{T}(r_4)\rangle_n\right)$$

- This 4-point function has been investigated in CFT but the analytic continuation remains challenging, even for free theories.
- An interesting limit is $\lim_{r_2\to r_3}\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\sim\tilde{\mathcal{T}}^2(r_3)$ where $\tilde{\mathcal{T}}^2$ is defined as the twist field associated to the cyclic permutation $j\mapsto j-2$. This field has very different properties depending on whether n is even or odd!

Logarithmic Negativity from Twist Fields

- The twist field approach has been used in the study of the LN of CFT [Calabrese, Cardy & Tonni'12'13'14]

Logarithmic Negativity from Twist Fields

$$\mathcal{E}[n] = \log\left(\varepsilon^{8\Delta_n}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\mathcal{T}(r_4)\rangle_n\right)$$

- This 4-point function has been investigated in CFT but the analytic continuation remains challenging, even for free theories.
- An interesting limit is $\lim_{r_2\to r_3}\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\sim\tilde{\mathcal{T}}^2(r_3)$ where $\tilde{\mathcal{T}}^2$ is defined as the twist field associated to the cyclic permutation $j\mapsto j-2$. This field has very different properties depending on whether n is even or odd!
- Calabrese et al. showed that (if $r_2=r_3=0$) then:

$$\mathcal{E}=\frac{c}{4}\log\left(\frac{r_1r_4}{r_1+r_4}\right)+\text{constant}$$

LN in Massive QFT: Adjacent Regions

- In our work we have studied two simple limits of the LN in a completely generic 1+1 dimensional QFT. Let us look at one of them:

LN in Massive QFT: Adjacent Regions

- In our work we have studied two simple limits of the LN in a completely generic 1+1 dimensional QFT. Let us look at one of them:
- *Adjacent regions* (one semi-infinite region): $r_3 \to r_2 := r$ and $r_4 \to \infty$ and we will choose $r_1 = 0$

$$\mathcal{E}_e^{\perp}[n] = \log\left(\varepsilon^{4\Delta_n + 4\Delta_{\frac{n}{2}}} \langle \mathcal{T}(0)\tilde{\mathcal{T}}^2(r)\rangle_n \langle \mathcal{T}\rangle_n\right)$$

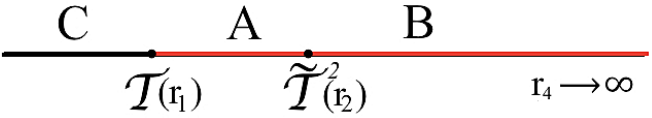
$2\Delta_{\frac{n}{2}}$ is the conformal dimension of $\mathcal{T}^2$ for n even.

LN in Massive QFT: Adjacent Regions

- In our work we have studied two simple limits of the LN in a completely generic 1+1 dimensional QFT. Let us look at one of them:
- *Adjacent regions* (one semi-infinite region): $r_3 \to r_2 := r$ and $r_4 \to \infty$ and we will choose $r_1 = 0$

$$\mathcal{E}_e^{\perp}[n] = \log\left(\varepsilon^{4\Delta_n + 4\Delta_{\frac{n}{2}}} \langle \mathcal{T}(0)\tilde{\mathcal{T}}^2(r)\rangle_n \langle \mathcal{T}\rangle_n\right)$$

$2\Delta_{\frac{n}{2}}$ is the conformal dimension of $\mathcal{T}^2$ for n even.



Results

- For adjacent regions, we found:

$$\mathcal{E}^{\perp} \overset{mr\to 0}{\sim} \frac{c}{4}\log(r/\varepsilon)$$
$$\overset{mr\gg 1}{=} -\frac{c}{4}\log(m\varepsilon)+\mathcal{E}_{\rm sat}-\frac{2}{3\sqrt{3}\pi}\sum_{\alpha}K_0(\sqrt{3}m_\alpha r)+O(e^{-Zmr})$$

with $Z>\sqrt{3}$, $m:=m_1$ the smallest mass in the spectrum, $\{m_\alpha\}$ the mass spectrum and $\mathcal{E}_{\rm sat}$ a universal saturation constant given by:

$$\mathcal{E}_{\rm sat}=2\log\left(m^{\frac{c}{8}}\langle\mathcal{T}\rangle_{\frac{1}{2}}\right)-\log(C_1) \qquad \text{and} \qquad C_1=\lim_{n\to 1}C_{\mathcal{T}\mathcal{T}}^{\mathcal{T}^2}$$

Remarks

Remarks

- The results are very simple and hold in this form for any 1+1 dimensional QFT (integrable or not)

Remarks

- The results are very simple and hold in this form for any 1+1 dimensional QFT (integrable or not)
- The technical reason for this is that they only depend on the pole structure of the twist field matrix elements and of the S-matrix, not on the details of the theory

Remarks

- The results are very simple and hold in this form for any 1+1 dimensional QFT (integrable or not)
- The technical reason for this is that they only depend on the pole structure of the twist field matrix elements and of the S-matrix, not on the details of the theory
- This means that the negativity (as the EE) can provide information about the mass spectrum of QFT

Remarks

- The results are very simple and hold in this form for any 1+1 dimensional QFT (integrable or not)
- The technical reason for this is that they only depend on the pole structure of the twist field matrix elements and of the S-matrix, not on the details of the theory
- This means that the negativity (as the EE) can provide information about the mass spectrum of QFT
- Numerical simulations in gapped and critical quantum spin chain models could be used to extract information about the VEVs, the structure constant and the mass spectrum of any 1+1 dimensional massive QFT

Remarks

- The results are very simple and hold in this form for any 1+1 dimensional QFT (integrable or not)
- The technical reason for this is that they only depend on the pole structure of the twist field matrix elements and of the S-matrix, not on the details of the theory
- This means that the negativity (as the EE) can provide information about the mass spectrum of QFT
- Numerical simulations in gapped and critical quantum spin chain models could be used to extract information about the VEVs, the structure constant and the mass spectrum of any 1+1 dimensional massive QFT
- Such numerical checks have been carried out for the EE [Levi, OCA & Doyon'12; Sirker et al.'14]

Remarks

- The results are very simple and hold in this form for any 1+1 dimensional QFT (integrable or not)
- The technical reason for this is that they only depend on the pole structure of the twist field matrix elements and of the S-matrix, not on the details of the theory
- This means that the negativity (as the EE) can provide information about the mass spectrum of QFT
- Numerical simulations in gapped and critical quantum spin chain models could be used to extract information about the VEVs, the structure constant and the mass spectrum of any 1+1 dimensional massive QFT
- Such numerical checks have been carried out for the EE [Levi, OCA & Doyon'12; Sirker et al.'14]
- In order to obtain these results, it was necessary to develop an approach to the analytic continuation in n.

Analytic continuation in QFT

- Sums of the form $\sum_{i=1}^{n/2} f(\{\theta\}, n) \mapsto \oint \cot(\pi z) f(\{\theta\}, z) dz$.

Analytic continuation in QFT

- Sums of the form $\sum_{i=1}^{n/2} f(\{\theta\}, n) \mapsto \oint \cot(\pi z) f(\{\theta\}, z) dz$.
- Uniqueness? Yes, up to Carlson's theorem! "two different analytic functions which do not grow very fast at infinity can not coincide at the integers"

Analytic continuation in QFT

- Sums of the form $\sum_{i=1}^{n/2} f(\{\theta\}, n) \mapsto \oint \cot(\pi z) f(\{\theta\}, z) dz$.
- Uniqueness? Yes, up to Carlson's theorem! "two different analytic functions which do not grow very fast at infinity can not coincide at the integers"
- The functions $f(\{\theta\}, n)$ have certain properties as $n \to 1$ which we can use.

Analytic continuation in QFT

- Sums of the form $\sum_{i=1}^{n/2} f(\{\theta\}, n) \mapsto \oint \cot(\pi z) f(\{\theta\}, z) dz$.
- Uniqueness? Yes, up to Carlson's theorem! "two different analytic functions which do not grow very fast at infinity can not coincide at the integers"
- The functions $f(\{\theta\}, n)$ have certain properties as $n \to 1$ which we can use.
- Starting at n large and approaching $n = 1$ poles of the functions $f(\{\theta\}, n)$ on the rapidities may cross the real line. Since the form factor expansion involves integration over the full real line over all rapidities it follows that the residues of these poles must be added!

Conclusion & Outlook

- Although a full computation of the LN for 1+1 dimensional QFTs remains challenging, we have shown that in particular limits it exhibits remarkable universality

Conclusion & Outlook

- Although a full computation of the LN for 1+1 dimensional QFTs remains challenging, we have shown that in particular limits it exhibits remarkable universality
- In particular, its leading large-r behaviour is fully determined by the mass spectrum of the QFT

Conclusion & Outlook

- Although a full computation of the LN for 1+1 dimensional QFTs remains challenging, we have shown that in particular limits it exhibits remarkable universality
- In particular, its leading large-r behaviour is fully determined by the mass spectrum of the QFT
- Work on measures of entanglement contributes to our understanding of the fundamental properties of ground states in QFT

Conclusion & Outlook

- Although a full computation of the LN for 1+1 dimensional QFTs remains challenging, we have shown that in particular limits it exhibits remarkable universality
- In particular, its leading large-r behaviour is fully determined by the mass spectrum of the QFT
- Work on measures of entanglement contributes to our understanding of the fundamental properties of ground states in QFT
- It also leads to interesting mathematical problems relating to the analytic continuation of functions of many complex variables