Entanglement Entropy of non-Unitary Quantum Field Theory

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Entangle This: Space, Time & Matter
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This talk is mainly based on the following works:

D. Bianchini, O.C.-A., B. Doyon, E. Levi and F. Ravanini, Entanglement Entropy of Non Unitary Conformal Field Theory, J. Phys. A48 04FT01 (2015).

D. Bianchini, O.C.-A. and B. Doyon, Entanglement Entropy of Non-Unitary Integrable Quantum Field Theory; arXiv:1502.03275

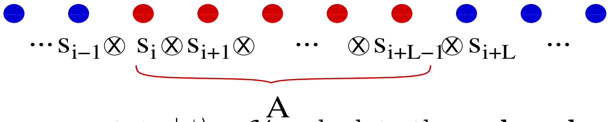
I will also often refer to our first paper on Entanglement Entropy where the connection to branch-point twist fields was introduced:

J.L. Cardy, O.C.-A. and B. Doyon, Form Factors of Branch-Point Twist Fields in Quantum Integrable Models and Entanglement Entropy, J. Stat. Phys. 130 129-168 (2008).

How to measure (or quantify) quantum entanglement?

In many-body quantum systems a popular measure of quantum entanglement is the entanglement entropy (EE):

- Choose a decomposition of the Hilbert space into a tensor product $\mathcal{H} = \mathcal{A} \otimes \mathcal{B}$. For instance:



- Given a pure state $|\psi\rangle \in \mathcal{H}$, calculate the **reduced density matrix**:

$$\rho_A = \mathrm{Tr}_{\mathcal{B}}(|\psi\rangle\langle\psi|) \in \mathrm{End}(\mathcal{A})$$

- There various kinds of entanglement entropy (EE) that we may define such as:

$$\begin{aligned} \text{Rényi} \quad &: \quad S_n(A) = \frac{1}{1-n} \log \mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \\ \text{von Neumann} \quad &: \quad S(A) = -\mathrm{Tr}_{\mathcal{A}}(\rho_A \log(\rho_A)) = S_1(A) \\ \text{“Replica Trick”} \quad &: \quad = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \end{aligned}$$

EE in 1+1-D (Unitary) Quantum Systems

- Exact results and determinant representations in free fermion chains [Peschel'04; Its, Jin, Korepin'05], connections to Painlevé equations [Casini, Fosco, Huerta'05; Casini, Huerta'05].
- Exact values in gapped integrable chains: infinite chain, semi-infinite region by CTM [Peschel'04; Ercolessi, Evangelisti, Ravanini'10; Ercolessi, Evangelisti, Franchini, Ravanini'11].
- Universal results at and near criticality: entanglement entropy diverges logarithmically with correlation length [Holzhey, Larsen, Wilczek'94; Vidal, Latorre, Rico, Kitaev'03; Latorre, Rico, Vidal'04; Calabrese, Cardy'04].
- Universal results at criticality for the entropy of excited states [Ibañez Berganza, Castillo Alcaraz & Sierra'11]and the EE of disconnected regions [Cardy, Calabrese, Tonni'09] and the negativity [Cardy, Calabrese & Tonni'12].
- Universal results for 1+1 dimension integrable models (beyond criticality) using a representation of the EE in terms of correlation functions of twist fields [Cardy, O.C.-A.,Doyon'08].

Important Results At and Near Critical Points

- Short distance (CFT): $0 \ll L \ll \xi$, logarithmic behavior [Holzhey, Larsen, Wilczek'94; Vidal, Latorre, Rico, Kitaev'03; Latorre, Rico, Vidal'04; Calabrese, Cardy'04]

$$S_n(A) \sim |\partial A| \frac{c(n+1)}{12n} \log(L/\epsilon)$$

where c is the central charge of the CFT and ϵ is a short-distance cut-off.

- Large distance (massive QFT): $0 \ll \xi \ll L$, saturation [Calabrese, Cardy'04]

$$S_n(A) \sim |\partial A| \frac{c(n+1)}{12n} \log(\xi\epsilon) + |\partial A|\, U_n + \text{exp. corrections}$$

At $n=1$, $|\partial A| = 2$ [Cardy, O.C.-A., Doyon'08; Doyon'09]

$$S_1(A) \sim \frac{c}{3} \log(\xi\epsilon) + 2U_1 - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3m_1 r}\right)$$

m_α is the mass spectrum, $m_1 \propto \xi^{-1}$ is the smallest mass.

Motivation & Central Question

- Suppose that we are given a pure state $|\psi\rangle$ and that we perform a computation of the EE.
- If we find logarithmic scaling we may deduce that the system is critical and we expect to be able to extract the central charge c of the critical point.
- What happens if $|\psi\rangle$ is the ground state of a critical system described by a non-unitary CFT? We find that we may just replace $c \to c_{\text{eff}} = c - 24\Delta$.
- Here c_{eff} is the effective central charge and Δ is the smallest conformal dimension of a primary field in the theory [Itzykson, Saleur & Zuber'86].
- For example for the Lee-Yang minimal model $c = -22/5$ and $\Delta = -1/5$ so $c_{\text{eff}} = 4/5$ [Fisher'78; Cardy'85].
- If we only know the state $|\psi\rangle$ we can not tell whether we are seeing c or c_{eff}. In general the EE will give us c_{eff}.
- We may tell unitary and non-unitary critical systems apart by studying the EE near criticality!

CFT results

Our main result is that the Rényi entropy of an interval of length r starting at the boundary is given by

$$S_n(A) \sim \frac{c_{\text{eff}}(n+1)}{12n} \log\left(\frac{r}{\epsilon}\right) \quad \text{for} \quad |\partial A| = 1$$

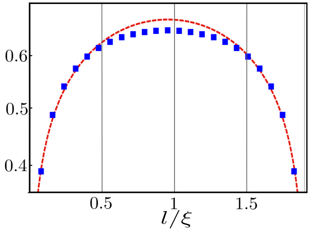
- Our derivation is close in spirit to [Holzhey, Larsen, Wilczek'94] where $\text{Tr}(\rho_A^n) = Z_n/(Z_1)^n$ is re-interpreted as a ratio of partition functions on the $\mathbb{Z}_n$ orbifold (replica theory) and the original CFT. In our derivation we made more extensive use of the algebraic structure of CFT.
- At critical points a geometric description, Riemann uniformization techniques and standard expressions for CFT partition functions is all that is needed.
- Near critical points, the scaling limit is described by massive QFT. CFT techniques fail.
- Thus if we want to go beyond criticality, a field theoretical approach to the EE becomes very powerful: twist fields

Numerical Evidence

Consider the standard Hilbert space structure on $(\mathbb{C}^2)^N$ and the non-hermitian Hamiltonian

$$H(\lambda, h) = -\frac{1}{2}\sum_{j=1}^{N}\left(\sigma_j^z + \lambda\sigma_j^x\sigma_{j+1}^x + ih\sigma_j^x\right)$$

- The Hamiltonian is PT-symmetric. There is a region of parameters where this PT-symmetry is unbroken. In that region it has a **real spectrum** [von Gehlen'91].
- In the thermodynamic limit, there is a **critical line** in the (λ, h)-plane (the line corresponding to PT-symmetry breaking) [von Gehlen'91; O.C.-A., Fring'09]. On this line, the spectrum and thermodynamic properties are those of the **Lee-Yang minimal CFT** [Fisher'78; Cardy'85].
- We have calculated the EE of the ground state for a finite (but long) chain and extracted $c_{\text{eff}} = 0.4$ in agreement with our general results.



- Here ℓ is the length of the block. The data are obtained by exact diagonalization for $\lambda = 0.9$ and $N = 24$. The dashed line is the fitting curve

$$\frac{\alpha}{3} \log\left(\frac{N}{\pi} \sin \frac{\pi L}{N}\right) + \beta$$

where the finite-volume form is used [Holzhey, Larsen, Wilczek'94; Calabrese, Cardy'04]. Fitting gives

$$\alpha = 0.4056, \qquad \beta = 0.3952.$$

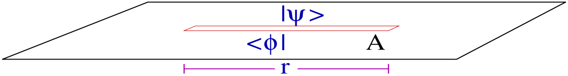
Twist Fields in QFT

- It has been known for some time that a "twist field" may be associated to the $\mathbb{Z}_n$ symmetry of an orbifolded CFT constructed as n cyclicly connected copies of a given CFT [Knizhnik'87]. The conformal dimension of such field $\mathcal{T}$ was also found by Knizhnik: $\Delta_{\mathcal{T}} = \frac{c}{24}\left(n - \frac{1}{n}\right)$.
- In the context of the investigation of the entanglement entropy a field of the same dimension was identified in [Calabrese, Cardy'04]. In this work, this field was interpreted as associated to a conical singularity in the complex plane.
- In 2008 we proposed [Cardy, O.C.-A. & Doyon'08] an interpretation of the fields found in [Calabrese & Cardy'04] as branch point twist fields.

Partition functions on multi-sheeted Riemann surfaces

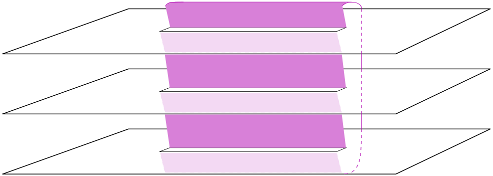
- For integer numbers n of replicas, in the scaling limit, $Z_n = \mathrm{Tr}_{\mathcal{A}}(\rho_A^n)$ is a partition function on a Riemann surface [Callan & Wilczek'94; Holzhey, Larsen & Wilczek'94; Calabrese & Cardy'04] ($\mathrm{Tr}_A(\rho_A)$ is the partition function of the original theory!):

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$



$$\mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\ \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n =$$



Twist Fields in QFT

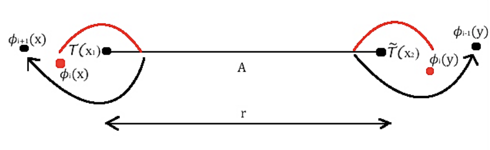
- Branch Point Twist Fields are characterized by the following commutation relations

$$\begin{aligned}
\phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\phi_{i+1}(y) \qquad x^1 > y^1, \\
\phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\phi_i(y) \qquad x^1 < y^1,
\end{aligned}$$

$$\begin{aligned}
\phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\phi_{i-1}(y) \qquad x^1 > y^1, \\
\phi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\phi_i(y) \qquad x^1 < y^1.
\end{aligned}$$

  where ϕ_i is a field of the original CFT living on copy i and $i = 1, \ldots, n$ and $n + i \equiv i$.

- Diagramatically:



EE and Correlation Functions in Unitary Theories

- In terms of twist field the EE may be written by employing the following relation [Calabrese & Cardy'04; Cardy, O.C.-A. & Doyon'08]

Entanglement Entropy in Unitary QFT

$$\mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \propto \epsilon^{\frac{c}{6}\left(n-\frac{1}{n}\right)}\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle.$$

where ϵ is a short-distance cut-off, c is the central charge and r is the length of region A.

- In CFT it is very easy to check that such representation indeed gives the expected formulae for the EE since

$$\epsilon^{4\Delta_{\mathcal{T}}}\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle = \left(\frac{\epsilon}{r}\right)^{4\Delta_{\mathcal{T}}} \;\Rightarrow\; S(A) \sim \frac{c}{3}\log\left(\frac{r}{\epsilon}\right)$$

- Saturation for large distances (massive QFT) is reached via

$$\lim_{r\to\infty} \epsilon^{4\Delta_{\mathcal{T}}}\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle = \epsilon^{\frac{c}{6}\left(n-\frac{1}{n}\right)}\langle \mathcal{T}\rangle^2 \;\Rightarrow\; S(A) \sim \frac{c}{3}\log(\epsilon m^{-1})+U$$

- Our CFT investigation has led us to conclude that for non-unitary theories the EE should rather be given by

Entanglement Entropy vs Correlators

$$\mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \propto \epsilon^{\frac{c_{\text{eff}}}{6}\left(n-\frac{1}{n}\right)} \frac{\langle :\mathcal{T}\phi:(r) :\tilde{\mathcal{T}}\phi:(0)\rangle}{\langle \phi(r)\phi(0)\rangle^n}.$$

where ϵ is a short-distance cut-off, c_{eff} is the effective central charge and r is the length of region A.

- The field $:\mathcal{T}\phi:$ is the leading term of the OPE of $\mathcal{T}$ and ϕ. It has conformal dimension $\Delta_{:\mathcal{T}\phi:} = \Delta_{:\tilde{\mathcal{T}}\phi:} = \Delta_{\mathcal{T}} + \frac{\Delta_\phi}{n}$ [Kac & Wakimoto'99; O.C.-A., Doyon & Levi'11; Levi'12].
- ϕ is the primary field of lowest (negative) conformal dimension (e.g. the CFT ground state is created by ϕ acting on the conformal vacuum).

Generalization to Massive QFT: the Lee-Yang Model

- We now want to compute the EE for a simple massive quantum field theory. The ideal model to look at is the Lee-Yang theory with S-matrix [Cardy & Mussardo'89]

$$S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)}.$$

- The underlying CFT is the Lee-Yang minimal model.
- Correlation functions of the fundamental field ϕ can be expressed in terms of form factors.
- Form factors were computed in [Zamolodchikov'91]. He was then able to compute $\langle \phi(r)\phi(0)\rangle$ with great precision and to match results to a perturbed CFT computation.

Form Factor Expansion

- Given a local operator $\mathcal{O}$ of a 1+1-dimensional QFT the two-point function

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\int \frac{d\theta_1}{2\pi}\cdots\int \frac{d\theta_k}{2\pi} |F_k^{\mathcal{O}}(\theta_1,\ldots,\theta_k)|^2 e^{-ix^\mu \sum_{i=1}^{k} p_\mu(\theta_i)}$$

  where $F_k^{\mathcal{O}}(\theta_1,\ldots,\theta_k) := \langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_k\rangle$ is the k-particle form factor of $\mathcal{O}$.

- In order to study the short distance behaviour it is more convenient to compute

$$\log\left(\frac{\langle \mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle}{\langle \mathcal{O}\rangle^2}\right) = \sum_{k=1}^{\infty} \frac{1}{k!}\int \frac{d\theta_1}{2\pi}\cdots\int \frac{d\theta_k}{2\pi} H_k^{\mathcal{O}}(\theta_1,\ldots,\theta_k) e^{-ix^\mu \sum_{i=1}^{k} p_\mu(\theta_i)}$$

  where H_k are related to F_k by simply matching terms in the two formulae.

  There are some subtleties in non-unitary theories!

CFT versus massive QFT

- Although at the critical point $\langle \phi(r)\phi(0)\rangle = r^{-4\Delta} = r^{4/5}$ when a mass is introduced the behaviour is rather

$$\langle \phi(r)\phi(0)\rangle/\langle\phi\rangle^2 = r^{-4\Delta}\left(1+\langle\phi\rangle C_{\phi\phi}^{\phi} r^{2\Delta}\right) + \text{corrections}$$

- At short distances for finite mass the two-point function goes as $\langle \phi(r)\phi(0)\rangle = \langle\phi\rangle C_{\phi\phi}^{\phi} r^{-2\Delta} = \langle\phi\rangle C_{\phi\phi}^{\phi} r^{2/5}$.
- This behaviour was well understood in [Zamolodchikov'91]: at the critical point the leading OPE contribution scales as $r^{-4\Delta} = r^{4/5}$ but in the massive theory $\langle\phi\rangle \neq 0$ so that the leading contribution to the "perturbed OPE" does not come from the identity field but from ϕ itself. This produces the term $r^{-2\Delta} = r^{2/5}$.

OPEs of the field $:\mathcal{T}\phi:$

- Interestingly, the same occurs to the correlator $\langle :\mathcal{T}\phi:(r):\tilde{\mathcal{T}}\phi:(0)\rangle$ in the massive (n-copy) theory. Its leading behaviour in perturbed CFT will be given by a term proportional to $\langle\phi\rangle^n \neq 0$ corresponding to the field $\phi_1\cdots\phi_n$. Thus $\langle :\mathcal{T}\phi:(r):\tilde{\mathcal{T}}\phi:(0)\rangle \propto r^{-4\Delta_{:\mathcal{T}\phi:}-2n\Delta}$.
- For example a zeroth order perturbed CFT computation for $n=2$ yields (here C_{ab}^c are the CFT structure constants)

$$\langle :\mathcal{T}\phi:(r):\tilde{\mathcal{T}}\phi:(0)\rangle/\langle :\mathcal{T}\phi:\rangle^2$$

$$= r^{-4\Delta_{:\mathcal{T}\phi:}}\left(1+2C_{:\mathcal{T}\phi::\tilde{\mathcal{T}}\phi:}^{\phi_1+\phi_2}r^{2\Delta}\langle\phi\rangle+C_{:\mathcal{T}\phi::\tilde{\mathcal{T}}\phi:}^{\phi_1\phi_2}r^{4\Delta}\langle\phi\rangle^2\right)+\cdots$$

$$= r^{\frac{3}{2}}\left(1-(6.2515...)(mr)^{-\frac{2}{5}}+(8.5055...)(mr)^{-\frac{4}{5}}\right)+\cdots$$

- Crucially, the ratio $\frac{\langle :\mathcal{T}\phi:(r):\tilde{\mathcal{T}}\phi:(0)\rangle}{\langle\phi(r)\phi(0)\rangle^n} \sim r^{-\frac{c_{\text{eff}}}{3}(n-\frac{1}{n})}$ so the EE in massive QFT still displays the correct behaviour (predicted by CFT) at short distances.

Form Factors of Twist Fields

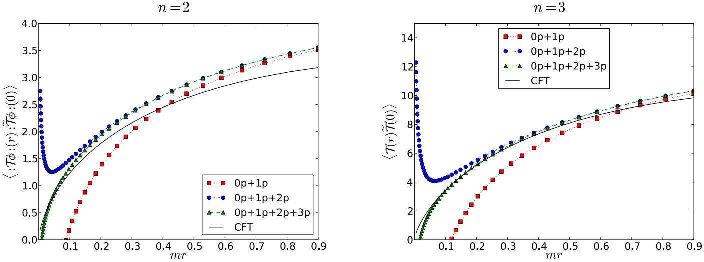
- In order to evaluate correlation functions we compute the form factors of twist fields [Cardy, O. C.-A. & Doyon'08]. This is a rather technical problem ... They can be obtained by solving a set of consistency conditions.
- Putting all these conditions together we find:

$$F_1^{\pm} = \frac{-i3^{1/4}\langle\mathcal{O}_\pm\rangle\left(\cos\left(\frac{\pi}{3n}\right) \pm 2\sin^2\left(\frac{\pi}{6n}\right)\right)}{\sqrt{2}n\sin\left(\frac{\pi}{3n}\right) f\left(\frac{2i\pi}{3}, n\right)}$$

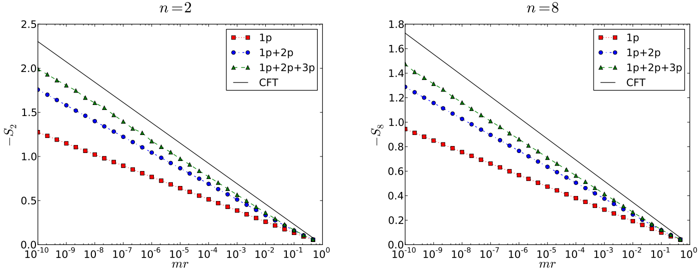
$$F_2^{11}(\theta) = \frac{\langle\mathcal{O}_\pm\rangle\sin\left(\frac{\pi}{n}\right)}{2n\sinh\left(\frac{i\pi-\theta}{2n}\right)\sinh\left(\frac{i\pi+\theta}{2n}\right)}\frac{F_{\min}^{11}(\theta,n)}{F_{\min}^{11}(i\pi,n)} + \frac{(F_1^{\pm})^2}{\langle\mathcal{O}_\pm\rangle}F_{\min}^{11}(\theta,n)$$

  where $F_{\min}^{11}(\theta)$ is a minimal solution to the FF equations which is proportional to a known function $f(\theta, n)$.
- Setting $n = 1$ gives either $F_1^- = 0$ or $F_1^+ = F_1^\phi$. This is strong indication that the FFs do indeed correspond to $\mathcal{O}_- = \mathcal{T}$ and $\mathcal{O}_+ =: \mathcal{T}\phi :$.

Numerical Evidence



- The figures show a comparison between form factors and perturbed CFT. They show good agreement for intermediate values of mr as expected.
- For very small mr the perturbed CFT results should be trusted whereas for large mr the form factor results should be the most accurate.
- The form factor results contain a further uncertainty since the values of $\langle \mathcal{T} \rangle$ and $\langle : \mathcal{T}\phi : \rangle$ are not known exactly.



- The figures above shown the Rényi entropy defined as

$$S_n(A) = \frac{1}{1-n} \log \left[\frac{\langle : \mathcal{T}\phi : (r) : \tilde{\mathcal{T}}\phi : (0) \rangle}{\langle \phi(r)\phi(0) \rangle^n} \right]$$

- The EE is plotted in logarithmic scale to make the proportionality to $\log(mr)$ apparent. Adding form factor contributions makes the slope of the lines get closer to the CFT prediction $S_n(A) = \frac{c_{\text{eff}}(n+1)}{6n} \log(mr)$.

Corrections to EE saturation at large regions

If we now consider

$$S = -\lim_{n\to 1}\frac{d}{dn}\epsilon^{\frac{c_{\text{eff}}}{6}\left(n-\frac{1}{n}\right)}\frac{\langle :\mathcal{T}\phi:(r):\tilde{\mathcal{T}}\phi:(0)\rangle}{\langle\phi(r)\phi(0)\rangle^n}$$

and we use a FF expansion

$$= -\frac{c_{\text{eff}}}{3}\log(m\epsilon) + U - \underbrace{\frac{2}{\pi f(\frac{2\pi i}{3},1)^2}\left(\frac{1}{\sqrt{3}}-\frac{13\pi}{108}\right)}_{0.0769782}K_0(mr)+\cdots$$

where

$$U = -\frac{d}{dn}\left(\frac{K_\phi^n}{K_{:\mathcal{T}\phi:}}\right)_{n=1}$$

and

$$K_\phi = m^{2\Delta}\frac{C_{\phi\phi}^{\phi}}{\langle\phi\rangle}, \quad K_{:\mathcal{T}\phi:} = m^{4\Delta-2n\Delta}\frac{C^{\phi_1\cdots\phi_n}_{:\mathcal{T}\phi::\tilde{\mathcal{T}}\phi:}\langle\phi\rangle^n}{\langle :\mathcal{T}\phi:\rangle^2}$$

The constants $K_{\mathcal{O}}$ are accessible from a form factor expansion [Babujian, Karowski'03].

Corrections to EE saturation at large regions

- The cut-off ϵ is chosen so as to have

$$\begin{aligned} S(r) &\sim -\frac{c_{\text{eff}}}{3}\log(m\epsilon)+U+o(1) && (mr\to\infty)\\ &\sim \frac{c_{\text{eff}}}{3}\log(r/\epsilon)+o(1) && (mr\to 0)\end{aligned}$$

- With this choice the constant U is universal in the sense that it can be computed purely in terms of fixed QFT quantities (e.g. VEVs and structure constants).
- For Lee-Yang we have found also further corrections as:

$$S(r)=-\frac{2}{15}\log(m\epsilon)+U-aK_0(mr)-\frac{be^{-2mr}}{\sqrt{2mr}}-\frac{ce^{-2mr}}{2mr}+O(\frac{e^{-2mr}}{(2mr)^{3/2}})$$

with $a=0.0769782..$, $b=0.326234..$ and $c=-0.0512159...$

Conclusions

- We have shown that the EE of non-unitary CFT satisfies similar scaling properties as for unitary theories with the replacement $c \to c_{\text{eff}}$.
- We have tested this result for a non-Hermitian spin chain model and a lattice model
- The description of the EE as a ratio of correlation functions appears to be consistent even beyond criticality. When a mass scale is added the CFT results are recovered as $mr \to 0$.
- Away from the critical point the EE saturates as for unitary models but the next-to-leading correction to saturation is different from that found for unitary 1+1-dimensional models.

- This means that examining the corrections to saturation of the EE for $L >> \xi$ may provide a method to identify non-unitary critical points!
- We would like first to test this prediction on the spin chain model mentioned earlier.
- Is there an "entropic" c_{eff}-theorem? [Casini, Huerta'06]
- What is the meaning of ratios such as?

$$\langle\langle \mathcal{O}(r)\tilde{\mathcal{O}}(0)\rangle\rangle := \frac{\langle : \mathcal{O}\phi : (r) : \tilde{\mathcal{O}}\phi : (0)\rangle}{\langle \phi(r)\phi(0)\rangle}$$

- Could they provide a clue to how to write down the "physical" correlators in non-unitary QFT?

A subtlety: right and left eigenvectors

- In the orbifold computation, we used Euclidean CFT. In Euclidean field theory, the vector constructed by the integration towards the infinite past is the right eigenvector $|\psi_R\rangle$, and that by the integration towards the infinite future the left eigenvector $\langle\psi_L|$. For non-hermitian Hamiltonians, these are generically **different**. So we are effectively evaluating

$$\mathrm{Tr}_{\mathcal{A}}\rho_A^n, \qquad \rho_A = \mathrm{Tr}_{\mathcal{B}}|\psi_R\rangle\langle\psi_L|$$

- At criticality they seem to be the same: $|\psi_R\rangle = |\psi_L\rangle$. This is because they are not only `PT`-symmetric, but also `P`-symmetric.

- In radial quantization $z = e^{i\eta+\tau}$ where η is space and τ is Euclidean time we represent $|\psi_R\rangle$ by the field $\phi(0)|0\rangle = \phi_+(0)\phi_-(0)|0\rangle$ (chiral decomposition). $z \mapsto \bar{z}, \phi_+ \mapsto \phi_-$ is Parity symmetry. Hence $\mathrm{P}|\psi_R\rangle = |\psi_R\rangle$ and $\mathrm{P}|\psi_L\rangle = |\psi_L\rangle$.
- In general we expect $T|\psi_R\rangle = |\psi_L\rangle$. Using PT invariance,

$$|\psi_R\rangle = \mathrm{PT}|\psi_R\rangle = \mathrm{P}|\psi_L\rangle = |\psi_L\rangle$$

  Checked numerically and from lattice model constructions.
- We believe this may be a feature that extends to the near critical behaviour.