



CITY UNIVERSITY
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Entanglement Entropy of non-Unitary Quantum Field Theory

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Beyond integrability: The mathematics and physics of
integrability and its breaking in low-dimensional strongly
correlated quantum phenomena

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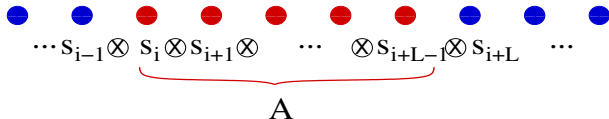
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$$\text{“Replica Trick”} : = -\lim_{n \rightarrow 1} \frac{d}{dn} \text{Tr}_{\mathcal{A}}(\rho_A^n)$$

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m_{α} is the mass spectrum, $m_1 \propto \xi^{-1}$ is the smallest mass.

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- Here c_{eff} is the effective central charge and Δ is the smallest conformal dimension of a primary field in the theory [Itzykson, Saleur & Zuber'86].
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- If we only know the state $|\psi\rangle$ we can not tell whether we are seeing c or c_{eff} . In general the EE will give us c_{eff} .
- **We may tell unitary and non-unitary critical systems apart by studying the EE near criticality!**

We have shown that the Rényi entropy of an interval of length L starting at the boundary is given by

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- At critical points a geometric description, Riemann uniformization techniques and standard expressions for CFT partition functions is all that is needed.
- Near critical points, the scaling limit is described by massive QFT. CFT techniques fail.
- Thus if we want to go beyond criticality, a field theoretical approach to the EE becomes very powerful: twist fields

Twist Fields in QFT

- It has been known for some time that a “twist field” may be associated to the \mathbb{Z}_n symmetry of an orbifolded CFT constructed as n cyclicly connected copies of a given CFT [Knizhnik'87]. The conformal dimension of such field \mathcal{T} was also found by Knizhnik: $\Delta_{\mathcal{T}} = \frac{c}{24} \left(n - \frac{1}{n} \right)$.

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- In 2008 we proposed [Cardy, O.C.-A. & Doyon’08] an interpretation of the fields found in [Calabrese & Cardy’04] as branch point twist fields.

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- In a replica QFT, Branch Point Twist Fields are characterized by the following commutation relations

$$\begin{aligned}\phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\phi_{i+1}(y) & x^1 > y^1, \\ \phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\phi_i(y) & x^1 < y^1,\end{aligned}$$

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where ϕ_i is a field of the original CFT living on copy i and $i = 1, \dots, n$ and $n + i \equiv i$.

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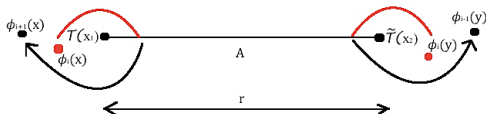
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- Diagrammatically:



- In terms of twist field the EE may be written by employing the following relation [Calabrese & Cardy'04; Cardy, O.C.-A. & Doyon'08]

Entanglement Entropy in Unitary QFT

$$\mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \propto \epsilon^{\frac{c}{6}(n-\frac{1}{n})} \langle \mathcal{T}(r) \tilde{\mathcal{T}}(0) \rangle.$$

where ϵ is a short-distance cut-off, c is the central charge and r is the length of region A .

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- From this description it is trivial to check that both the CFT results ($r \ll \xi$) [log scaling] and the QFT results ($\xi \ll r$) [saturation] may be recovered.
- This formulation also allows us to find sub-leading corrections to saturation (FF approach). This is what makes twist fields a powerful tool.

- Our CFT investigation has led us to conclude that for non-unitary theories the EE should rather be given by

Entanglement Entropy vs Correlators

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- ϕ is the primary field of lowest (negative) conformal dimension (e.g. the CFT ground state is created by ϕ acting on the conformal vacuum).

- We now want to compute the EE for a simple massive quantum field theory. The ideal model to look at is the Lee-Yang theory with S -matrix [Cardy & Mussardo'89]

$$S(\theta) = \frac{\tanh \frac{1}{2} \left(\theta + \frac{2\pi i}{3} \right)}{\tanh \frac{1}{2} \left(\theta - \frac{2\pi i}{3} \right)}.$$

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- Correlation functions of the fundamental field ϕ can be expressed in terms of form factors.
- Form factors were computed in [Zamolodchikov'91]. He was then able to compute $\langle \phi(r)\phi(0) \rangle$ with great precision and to match results to a perturbed CFT computation.

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- Setting $n = 1$ gives either $F_1^- = 0$ or $F_1^+ = F_1^{\phi}$. This is strong indication that the FFs do indeed correspond to $\mathcal{O}_- = \mathcal{T}$ and $\mathcal{O}_+ =: \mathcal{T}\phi$.:

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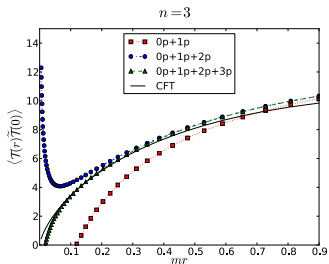
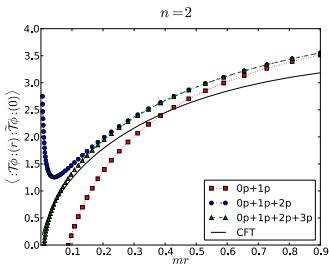
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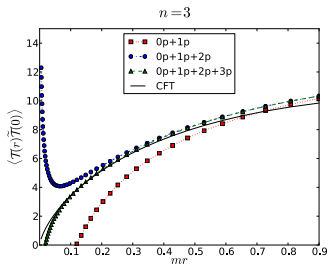
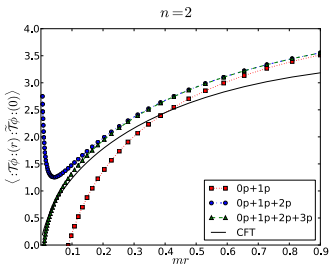
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Numerical Evidence

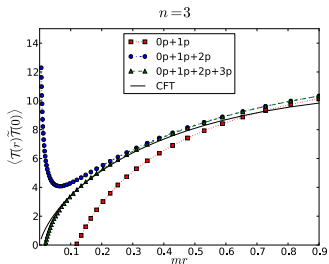
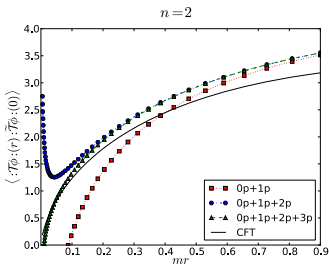


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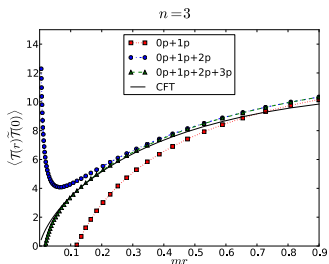
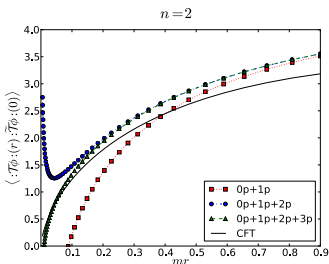
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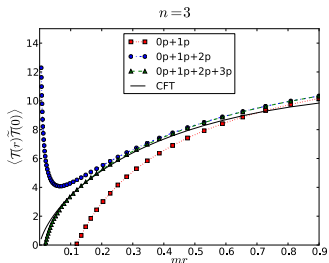
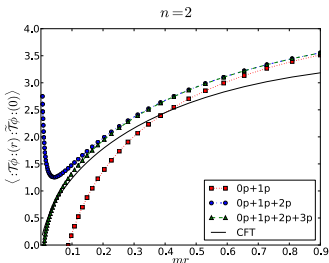
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- For very small mr the perturbed CFT results should be trusted whereas for large mr the form factor results should be the most accurate.
- The form factor results contain a further uncertainty since the values of $\langle \mathcal{T} \rangle$ and $\langle : \mathcal{T} \phi : \rangle$ are not known exactly.

Corrections to EE saturation at large regions

If we now consider

$$S(r) = - \lim_{n \rightarrow 1} \frac{d}{dn} \epsilon^{\frac{c_{\text{eff}}}{6} (n - \frac{1}{n})} \frac{\langle : \mathcal{T} \phi : (r) : \tilde{\mathcal{T}} \phi : (0) \rangle}{\langle \phi(r) \phi(0) \rangle^n}$$

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- Could they provide a clue to how to write down the “physical” correlators in non-unitary QFT?