Entanglement Entropy in 1+1 Dimensional QFT

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Overview of the talk

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- Conclusions and Outlook

Our contribution so far...

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- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.
  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right) \quad , \qquad \langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$

- What is special: Bell's inequality says that this cannot be described by **local variables**.
- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

$$|\hat{\psi}\rangle = \frac{1}{2} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right)$$

$$= \frac{1}{2} \left(|\uparrow\rangle + |\downarrow\rangle \right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle \right)$$

- This is particular to **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ , \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

  (for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).
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- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement?

- Let us consider a spin chain of length N, subdivided into regions A and $\bar{A}$ of lengths L and $N - L$

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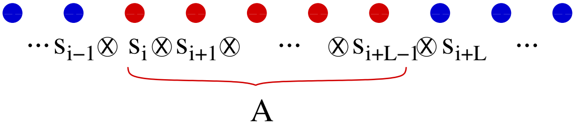

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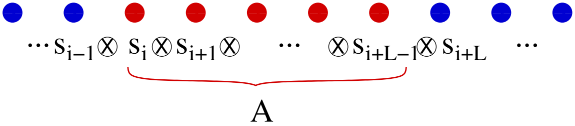

then we define

Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

$|\Psi\rangle$ ground state and ρ_A the reduced density matrix.

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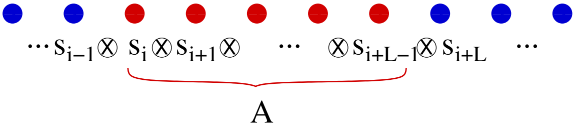

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- Other entropies may also be defined such as

Other Entropies

$$S_A^{\text{Rényi}} = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}, \qquad S_A^{\text{Tsallis}} = \frac{1-\mathrm{Tr}_A(\rho_A^n)}{n-1}$$

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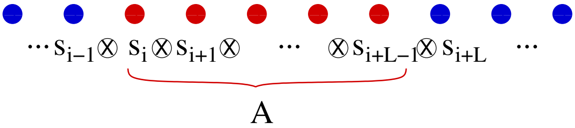

Replica Trick

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_A(\rho_A^n)$$

- For general QFTs the "replica trick" naturally leads to the notion of replica theories on multi-sheeted Riemann surfaces $\Rightarrow$ interpretation of $\mathrm{Tr}_A(\rho_A^n)$

Motivation

The study of the EE of extended quantum systems is a popular area of research in various areas:

- **Quantum Information:** The EE quantifies the amount of "surprise" that a sub-part of a system finds when discovering it is correlated to the rest of the system. Therefore, entanglement entropy is a *bona fide* measure of the correlations in the system [Latorre & Riera, Review'09]

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- **Quantum Field Theory:** The EE can be defined for any QFT (operator independent). It provides "universal" information about quantum systems/quantum states. [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Latorre, Rico & Kitaev'03; Latorre, Rico & Vidal'04; Calabrese & Cardy '04; J.L. Cardy, O.C-A & B. Doyon'08]

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- **Holography (AdS/CFT Correspondence):** The EE in the CFT is given by the area of a certain extremal surface in the bulk (AdS) [Ryu and Takayanagi'06]

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- This information does not depend on the correlation functions of any fields (just on the state).
- Near critical points (e.g. Conformal Field Theory) it displays universal behaviour.
- In the context of high energy physics much of the motivation to study the entanglement entropy has come from its behaviour at quantum critical points [Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04]:

$$S_m \sim \frac{c}{3} \log m \quad \Rightarrow \quad \text{information about the CFT}$$

e.g. the EE of a subsystem of length m diverges logarithmically. The proportionality constant c is called the central change. It uniquely characterises the CFT.

Example: the Ising model

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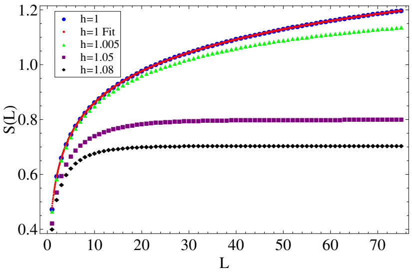

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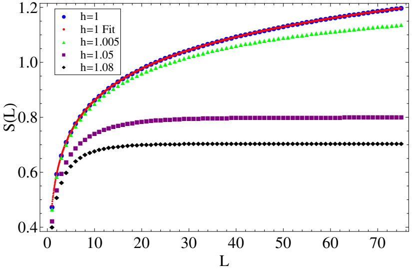

- $S(L) = \frac{0.500003}{3}\log L + 0.478551$ for $h = 1$. For $h > 1$ saturation is reached [Vidal, Latorre, Rico & Kitaev'03].

More motivation

- The entanglement entropy can reveal a great amount of information about the CFT (not just c). For example, the entropy of excited states [Ibañez Berganza, Castillo Alcaraz & Sierra'11] and the negativity [Cardy, Calabrese & Tonni'12] both encode information about the conformal dimensions of primary fields.

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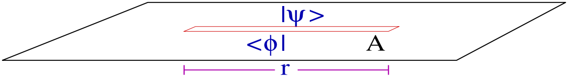
- In theories with boundaries, the entropy provides information about the degeneracy of the ground state (g-function)[Afleck & Ludwig'91; Calabrese & Cardy'04; OCA & Doyon'09].
- The entanglement entropy can also provide very interesting information about the state in cases when there is no conformal critical point [Ercolessi, Evangelisti, Franchini & Ravanini'10; Popkov & Salerno'04; Popkov, Salerno & Schütz'05], especially about the "geometry" of the state [OCA & Doyon'11;12].

Partition functions on multi-sheeted Riemann surfaces

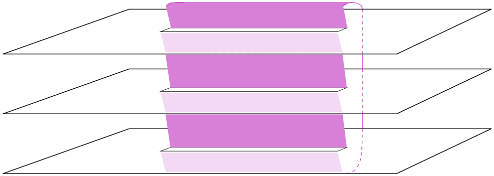
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04] ($\mathrm{Tr}_A(\rho_A)$ is the partition function of the original theory!):

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n =$$



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$$z = g(\omega) = \left(\frac{\omega - x_1}{\omega - x_2}\right)^{1/n} \quad \text{with} \quad z \in \mathbb{C}^2 \quad \text{and} \quad \omega \in \mathcal{M}_n$$

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- The problem with this approach is that it relies heavily on conformal symmetry. It is not applicable to systems which are not critical (e.g. have a mass scale).
- The is another point of view which allows for generalisation: employing twist fields.

Twist Fields I

- For general 1+1 dimensional QFT we have found [Cardy, OCA & Doyon'08] that the entropy may be expressed in terms of a two-point function of twist fields:

$$Z_n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0) \tilde{\mathcal{T}}(r) \rangle_n \ , \quad S_A = - \lim_{n \to 1} \frac{d}{dn} Z_n$$

where D_n is a normalisation constant, and d_n is the scaling dimension of $\mathcal{T}$ [Knizhnik'87; Calabrese & Cardy'04]:

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- Large distance: $0 \ll \xi \ll L$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \sim \langle \mathcal{T}\rangle_n^2 \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m\varepsilon) + U$$

Twist Fields II

- The Twist Fields are defined through very general commutation relations with the fundamental field of the model:

$$\begin{aligned}
\Psi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Psi_{i+1}(y) \qquad x^1 > y^1, \\
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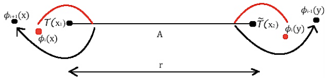
- Diagramatically:



Entropy from Form Factors

- The two-point function of branch-point twist fields can be decomposed into the *in*-basis, giving a large-distance expansion:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathbb{R}^n} = \langle \text{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\text{vac}\rangle =$$

$$\sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k=1}^{n}\int \frac{d\theta_1\cdots d\theta_k}{(2\pi)^k}\,|F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)|^2 e^{-r\sum\limits_{i=1}^{k} m_{\mu_i}\cosh\theta_i}$$

where

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- Typically the expansion is rapidly convergent in k for large r (short-distance expansion).

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- The two-point function of branch-point twist fields can be decomposed into the *in*-basis, giving a large-distance expansion:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathbb{R}^n} = \langle \text{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\text{vac}\rangle =$$

$$\sum_{k=0}^{\infty}\sum_{\mu_1,\ldots,\mu_k=1}^{n}\int \frac{d\theta_1\cdots d\theta_k}{(2\pi)^k}\,|F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)|^2 e^{-r\sum\limits_{i=1}^{k} m_{\mu_i}\cosh\theta_i}$$

where

$$F_k^{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k) = \langle \text{vac}|\mathcal{T}(0)|\theta_1,\ldots,\theta_k\rangle^{in}_{\mu_1,\ldots,\mu_k}$$

are the k-particle form factors of the twist-field $\mathcal{T}$.

- Typically the expansion is rapidly convergent in k for large r (short-distance expansion).
- These form factors can be computed as the solutions to a set of consistency equations which we formulated in our first paper [Cardy, OCA & Doyon'08].

Scaling behaviour of the von Neumann entropy

Let r be the length of region A in a massive 1+1 dimensional QFT, then

- Short distance: $0 \ll r \ll \xi$, *logarithmic divergency*

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- U is a universal constant which can be computed in QFT. m_1 is the mass of the lightest particle in the theory.

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- m_α are the particles's masses. The form of the correction follows from QFT techniques.

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- Taking $h \to 1$ and $L \to \infty$ simultaneously, with $\frac{L}{\xi} =: m_1 r$ fixed, we should recover the QFT saturation behaviour

Entropy of Gapped Chain

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Away from criticality: Next-to-leading order correction

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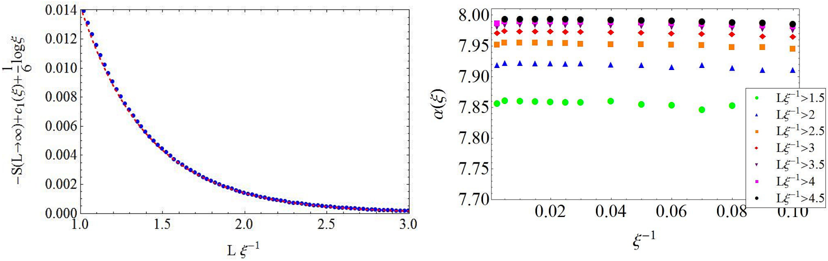

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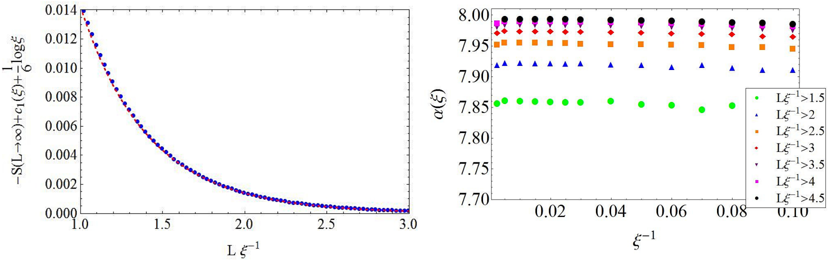

- Numerical results are fitted to the function $1/8K_0(2L/\xi)$ with very good agreement.
- $\alpha(\xi)$ is the inverse of the coefficient of the Bessel function (expected to be 8 at the critical point). It approaches the value 8 for increasing ratios L/ξ.

Conclusions

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- The EE encapsulates information about the particle spectrum of non-critical theories in 1+1 dimensions.
- Many open problems remain in this area which is still dominated by the investigation of critical systems.
- A natural next step is to look at other measures of entanglement which are more natural for mixed states (e.g. negativity), consider the EE of excited states and/or disconnected regions.