CITY UNIVERSITY LONDON

Thermodynamic Bethe ansatz for non-equilibrium steady states in integrable QFT

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It builds on previous results for CFT which have been discussed by Benjamin Doyon in a previous talk.

Overview of the talk

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- Conclusions

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TBA: Particles on a Trip Around the World

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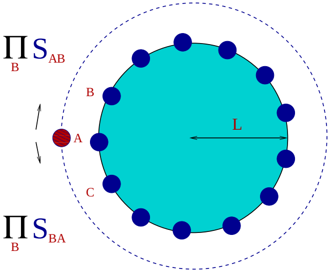

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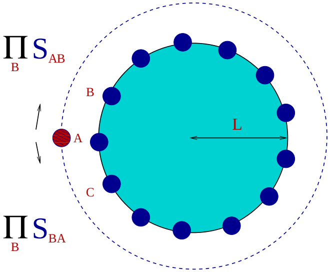

$$e^{iLM_A \sinh\theta_A} \prod_{A=1}^{N} S_{AB}(\theta_A - \theta_B) = 1$$

$$LM_A \sinh\theta_A + \sum_{B\neq A} \delta_{AB}(\theta_A - \theta_B) = 2\pi n_A, \quad A = 1, \cdots, n$$

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$$\epsilon_A(\theta) = M_A \beta \cosh\theta - \sum_A \varphi_{AB} * L_B(\theta)$$

- Here $\epsilon_A(\theta)$ are the pseudo-energies, $\beta = 1/T$, $\varphi_{AB} = -i \frac{d \ln(S_{AB}(\theta))}{d\theta}$, $L_B(\theta) = \ln(1 + e^{-\epsilon_B(\theta)})$ and $*$ indicates convolution $f * g(\theta) := \frac{1}{2\pi} \int f(\theta - \theta') g(\theta') d\theta'$.

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- The free energy is

$$f(\beta) = -\frac{\beta}{2\pi}\sum_{A=1}^{n} M_A \int_{-\infty}^{\infty} d\theta L_A(\theta)\cosh\theta := -\frac{\pi\beta^2 c_{\text{eff}}(\beta)}{6}.$$

TBA and CFT

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- In the UV limit $\lim_{\beta\to 0} c_{\text{eff}}(\beta) = c_{\text{eff}} := c - 24\Delta$ where c_{eff} is the effective central charge, c is the central charge and Δ is the lowest conformal dimension in the Kac’s table of the underlying CFT.

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- Other CFT properties may be recovered from the TBA equations. For example, when expressed in terms of Y-systems the latter exhibit periodicities which have been related to the dimension of the perturbing field [Zamolodchikov'91].

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  $W_A(\theta) = M_A \beta_l \cosh\theta$ for $\theta > 0$ and $W_A(\theta) = M_A \beta_r \cosh\theta$ for $\theta < 0$.
- This generalization follows from fundamental properties of integrable systems, especially that in the infinite past and future, states become well-separated, well-defined collections of wave packets behaving like free particles [Doyon'12]

The NESS energy current

- The current is a measure of energy transfer from the hot to the cold reservoir. In the TBA context this naturally corresponds to a derivative of the free energy:

$$J(\beta_l, \beta_r) = \left. \frac{df^a(\beta_l, \beta_r)}{da} \right|_{a=0} = \frac{1}{2\pi} \sum_A \int d\theta \frac{M_A \cosh\theta x_A(\theta)}{1 + e^{-\epsilon_A(\theta)}},$$

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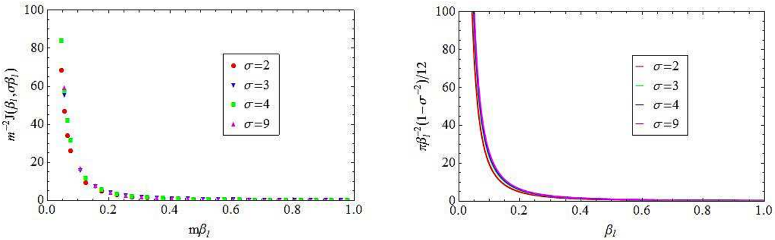
- $f^a(\beta_l,\beta_r)$ is the free energy, as defined earlier. The functions $x_A(\theta) = \left.\frac{d\epsilon_A(\theta)}{da}\right|_{a=0}$ can be obtained by solving

$$x_A(\theta) = M_A \sinh\theta + \sum_B \left(\varphi_{AB} * \frac{x_B}{1+e^{\epsilon_B}}\right)(\theta)$$

Current and L-functions: sinh-Gordon model



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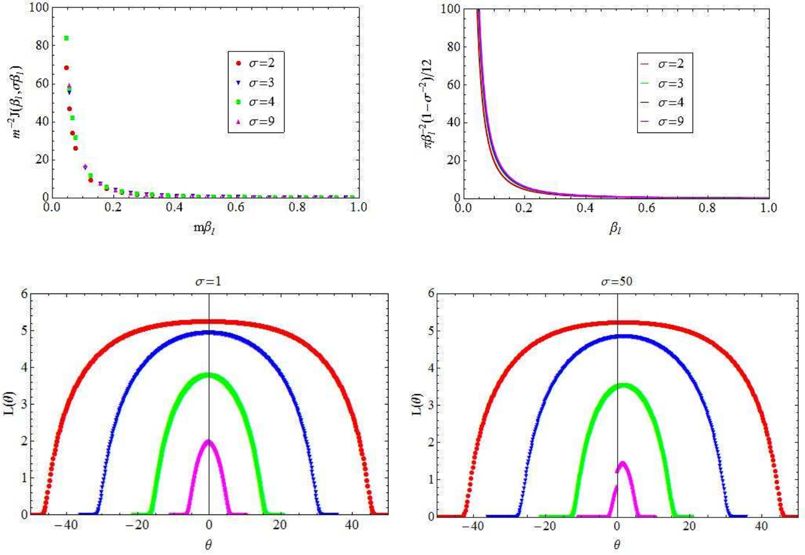

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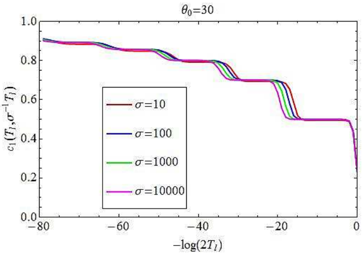
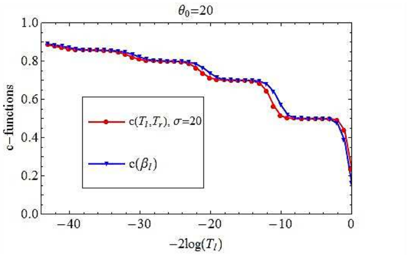
- From examples, it behaves as a new c-function!
- We can define a whole family of such functions if we also examine the current's cumulants.





Energy current as a Poisson process

- It has been shown [Bernard & Doyon'13] that the scaled cumulants generating function $F(z)$ in a system with pure energy transmission is given by

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- It is a superposition of Poisson processes for every energy E representing jumps towards the right or left with Maxwell-Boltzman factors $e^{\mp \beta_{l,r} E}$.
- This interpretation also holds for massive free theories. Numerical analysis suggests that it may also hold for other non-trivial QFTs.

Additivity/Non-additivity of the current: examples

- In CFT the current has the form $J(\beta_l, \beta_r) = f(\beta_l) - f(\beta_r)$. Therefore it satisfies the additivity property

$$P(\beta, \sigma) := \frac{J(\beta, \sigma\beta) + J(\sigma\beta, \sigma^2\beta) + J(\sigma^2\beta, \beta)}{J(\beta, \sigma^2\beta)} = 0$$

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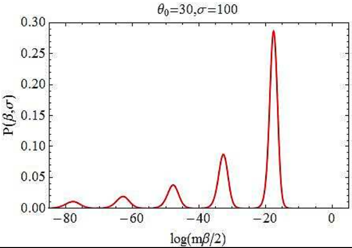
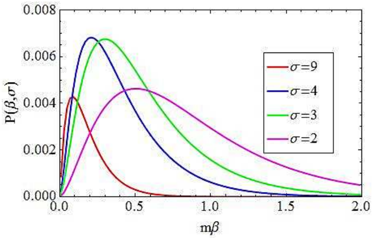
- Question: does this still hold for massive integrable QFT?
- Answer: It seems from our numerics and some perturbative calculations that the answer is <u>NO</u>. This can be seen quite strikingly in the roaming trajectories model





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- The monotonicity properties of these c-function impose constraints on the growth of the cumulants with a change in mass scale.
- We have found numerical evidence for an interpretation of the current flow as a Poisson process both in CFT and QFT.
- We have found evidence that the additivity of the current in not preserved in QFT. This is unlike results [Karrasch, Ilan & Moore'12] although they consider the non-universal, higher temperature regime of gapless chains.

Future directions

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- Prove that the functions $c_n(T_l, T_r)$ are c-functions.
- Carry out TBA numerics for non-diagonal theories (e.g. sine-Gordon model).