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Universal Features of the Negativity of 1+1 dimensional Quantum Field Theories

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- Throughout the talk I will also refer to some previous work, especially our first paper on the subject:

  John L. Cardy, Olalla Castro-Alvaredo and Benjamin Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, J. Stat. Phys. 130 (2008) 129-168.

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- What is special: Bell's inequality says that this cannot be described by **local variables**.
- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

$$|\hat{\psi}\rangle = \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

$$= \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right)$$

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- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
- The bi-partite entanglement entropy [Bennett et al.'96] and the logarithmic negativity [Vidal & Werner'01; Plenio'05] are good measures of entanglement according to these properties

Bi-partite Entanglement Entropy (EE)

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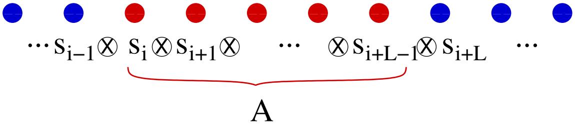

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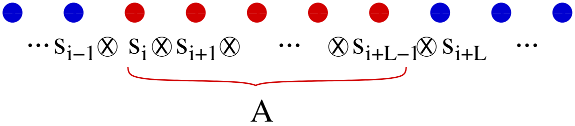

then we define

Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

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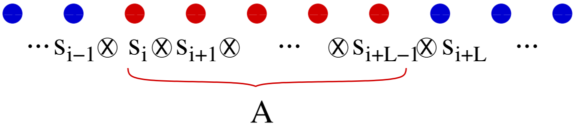

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Other Entropies

$$S_A^{\text{Rényi}} = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}, \qquad S_A^{\text{Tsallis}} = \frac{1-\mathrm{Tr}_A(\rho_A^n)}{n-1}$$

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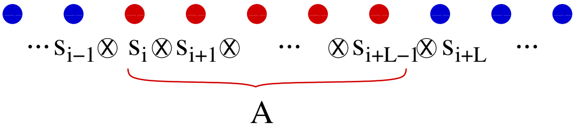

Replica Trick

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$$

- For general QFTs the "replica trick" naturally leads to the notion of replica theories on multi-sheeted Riemann surfaces $\Rightarrow$ interpretation of $\mathrm{Tr}_A(\rho_A^n)$

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$$S(L) \sim \frac{c_{\text{eff}}}{3} \log L \quad \Rightarrow \quad \text{information about the CFT}$$

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- Computing the EE is now the most efficient numerical approach to classifying critical points!

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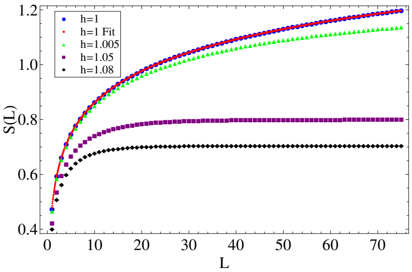

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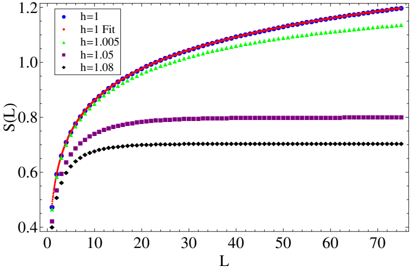

- $S(L) = \frac{0.500003}{3}\log L + 0.478551$ for $h = 1$. For $h > 1$ saturation is reached [Vidal, Latorre, Rico & Kitaev'03; Levi, OCA, Doyon'12].

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- $|\Psi\rangle$ is the state of the whole system (for pure states)

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- There is also a "replica" approach to the computation of the negativity [Calabrese, Cardy & Tonni'12]:

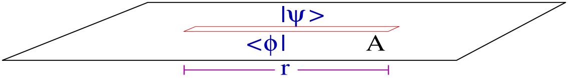
Logarithmic Negativity from the Replica Trick

$$\mathcal{E}[n] = \log \mathrm{Tr}_{A\cup B}(\rho_{A\cup B}^{T_B})^n \quad \text{then} \quad \mathcal{E} = \lim_{n\to 1} \mathcal{E}_e[n]$$

where $\mathcal{E}_e[n]$ means the function $\mathcal{E}[n]$ for n even. This limit requires analytic continuation from n even to $n = 1$

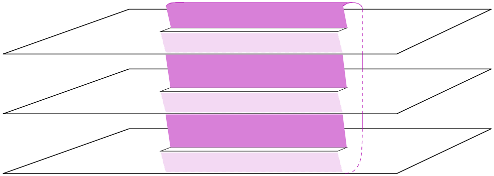
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface [Callan & Wilczek '94; Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04] ($\mathrm{Tr}_A(\rho_A)$ is the partition function of the original theory!):

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n =$$



Branch Point Twist Fields

- For general 1+1 dimensional QFT we have found [Calabrese, Cardy'04; Cardy, OCA & Doyon'08] that the EE may be expressed in terms of a two-point function of twist fields:

$$Z_n = D_n \varepsilon^{4\Delta_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

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  where D_n is a normalisation constant, and Δ_n is the conformal dimension of $\mathcal{T}$ [Knizhnik'87; Dixon et al.'87; Calabrese & Cardy'04]:

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- Large distance: $0 \ll \xi \ll r$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \sim \langle \mathcal{T}\rangle_n^2 \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m\varepsilon) + U$$

Main Properties of Twist Fields

- The Twist Fields are defined through very general commutation relations with the fundamental field of the model [Cardy, OCA & Doyon'08]:

$$\begin{aligned}
\Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_{i+1}(y) \qquad x^1 > y^1, \\
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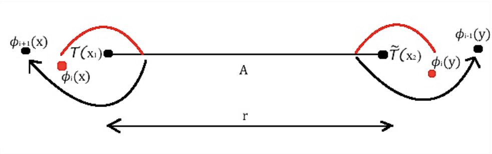
- Diagramatically:



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- An interesting limit is $\lim_{r_2\to r_3}\tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3)\sim\tilde{\mathcal{T}}^2(r_3)$ where $\tilde{\mathcal{T}}^2$ is defined as the twist field associated to the cyclic permutation $j\mapsto j-2$.

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Logarithmic Negativity from Twist Fields

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- This 4-point function has been investigated in CFT but the analytic continuation remains challenging, even for free theories.
- An interesting limit is $\lim_{r_2\to r_3} \tilde{\mathcal{T}}(r_2)\tilde{\mathcal{T}}(r_3) \sim \tilde{\mathcal{T}}^2(r_3)$ where $\tilde{\mathcal{T}}^2$ is defined as the twist field associated to the cyclic permutation $j \mapsto j-2$. This field has very different properties depending on whether n is even or odd!
- Calabrese et al. showed that (if $r_2 = r_3 = 0$) then:

$$\mathcal{E} = \frac{c}{4}\log\left(\frac{r_1 r_4}{r_1+r_4}\right) + \text{constant}$$

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$$\mathcal{E}_e^{\perp}[n] = \log \left(\varepsilon^{4\Delta_n + 4\Delta_{\frac{n}{2}}} \langle \mathcal{T}(0) \tilde{\mathcal{T}}^2(r) \rangle_n \langle \mathcal{T} \rangle_n \right)$$

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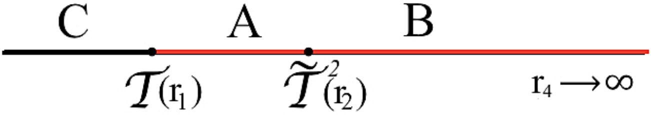

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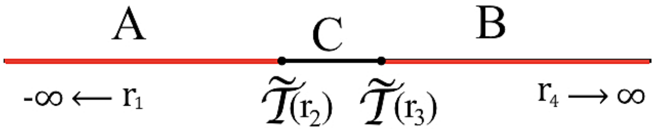

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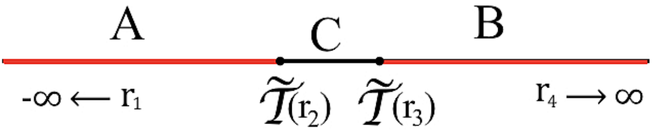

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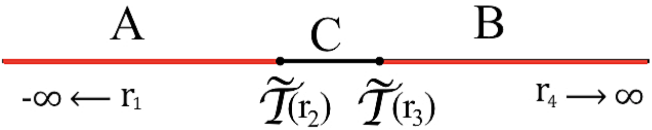

- Our aim is to investigate the leading contribution to these functions for large r. This can be accessed from the one and two-particle form factors of twist fields.

- For adjacent regions, we found:

$$\mathcal{E}^{\perp} \overset{mr\to 0}{\sim} \frac{c}{4}\log(r/\varepsilon)$$
$$\overset{mr\gg 1}{=} -\frac{c}{4}\log(m\varepsilon)+\mathcal{E}_{\rm sat}-\frac{2}{3\sqrt{3}\pi}\sum_{\alpha}K_0(\sqrt{3}m_\alpha r)+O(e^{-Zmr})$$

with $Z>\sqrt{3}$, $m := m_1$ the smallest mass in the spectrum, $\{m_\alpha\}$ the mass spectrum and $\mathcal{E}_{\rm sat}$ a universal saturation constant given by:

$$\mathcal{E}_{\rm sat} = 2\log\left(m^{\frac{c}{8}}\langle\mathcal{T}\rangle_{\frac{1}{2}}\right)-\log(C_1) \qquad \text{and} \qquad C_1=\lim_{n\to 1} C_{\mathcal{T}\mathcal{T}}^{\mathcal{T}^2}$$

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$$\overset{mr\gg 1}{=} \frac{1}{2\pi^2}\sum_{\alpha}(m_\alpha r)^2\left[K_0(m_\alpha r)^2 + \frac{K_0(m_\alpha r)K_1(m_\alpha r)}{m_\alpha r} - K_1(m_\alpha r)^2\right]$$

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- Such numerical checks have been carried out for the EE [Levi, OCA & Doyon'12; Sirker et al.'14]

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- Consider the case of adjacent regions. At short distances we have
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and so

$$\lim_{n\to 1} \log\left(\varepsilon^{4\Delta_n+4\Delta_{\frac{n}{2}}}\langle \mathcal{T}(0)\tilde{\mathcal{T}}^2(r)\rangle_n\langle \mathcal{T}\rangle_n\right) = -\frac{c}{4}\log(r/\varepsilon)+\log(C_1)$$

- Since ε is a non-universal cut-off we can also redefine ε to absorbe the constant C_1

Derivation: large-r

- For large r on the other hand we can use QFT factorization and we have

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- Upon redefinition of the cut-off $\varepsilon\to\varepsilon/(C_1)^{\frac{4}{c}}$ the saturation value above becomes (as anticipated)

$$\frac{c}{4}\log(m\varepsilon)+\mathcal{E}_{\rm sat}=\frac{c}{4}\log(m\varepsilon)+2\log(m^{-\frac{c}{8}}\langle\mathcal{T}\rangle_{\frac{1}{2}})-\log(C_1)$$

- Exponentially decaying corrections to this saturation can be obtained by using a form factor expansion of the two-point function. I will illustrate this with a simple example:

The free Boson

- For the free Boson it is known that:

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$$F(\theta_1-\theta_2, n) = \langle 0|\mathcal{T}|\theta_1\theta_2\rangle_{11} = \frac{\sin\frac{\pi}{n}}{2n\sinh\left(\frac{i\pi+\theta_1-\theta_2}{2n}\right)\sinh\left(\frac{i\pi-\theta_1+\theta_2}{2n}\right)}$$

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- The first non-trivial correction to the two point function $\langle\mathcal{T}(0)\tilde{\mathcal{T}}^2(r)\rangle_n$ comes from the two particle form factor and is given by the sum

$$n\sum_{j=0}^{\frac{n}{2}-1} F(\theta+4\pi i j, n)F(\theta+2\pi i j, \frac{n}{2})$$
$$= \frac{n}{2}F(\frac{i\pi-3\theta}{2}, \frac{n}{2})\tan\left(\frac{i\theta+\pi}{4}\right) + nF(2i\pi-3\theta, n)\tan\left(\frac{i\theta}{2}\right) - (\theta\to-\theta)$$

The free Boson (continued)

- The contribution to the negativity of the sum above is proportional to

$$n\int_{-\infty}^{\infty}\left(F(\frac{i\pi+3\theta}{2},\frac{n}{2})\tan\left(\frac{i\theta+\pi}{4}\right)+2F(2i\pi+3\theta,n)\tan\left(\frac{i\theta}{2}\right)\right)K_0(2mr\cosh\frac{\theta}{2})$$

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- This analysis can be generalized to any 1+1 d QFT

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- We hope our work will shed light on how to perform the required analytic continuation correctly, an issue which remains unresolved for CFT