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Entanglement Entropy and QFT: Form Factor Approach

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Overview of the talk

- Entanglement Entropy from Twist Fields
- Short and Long Distance Behaviours
- Correlation Functions from Form Factors
- Form Factor Program for Twist Fields
- Analytic Continuation
- Consistency Checks
- New Predictions for Quantum Field Theory
- Numerical tests
- Conclusions and Outlook

Entanglement Entropy from Twist Fields

- In Conformal Field Theory (CFT) the Entanglement Entropy (EE) may be computed either by:
  - ⋆ using conformal maps and uniformization theorem to map the partition function on a complicated Riemann surface to a partition function on the complex plane
  - ⋆ using correlation functions and twist fields which are completely determined by CFT (at least for two-points)
- The first approach does not extend beyond critical systems, as it uses conformal maps/symmetry
- However *it is possible to define correlation functions in any Quantum Field Theory* (QFT) and so expressing the EE in terms of correlation functions of twist fields provides a method which may be extended beyond CFT

Entanglement Entropy from Twist Fields

- This is the main reason why in 2007 we decided to take this approach as our starting point to try and extend the vast knowledge about EE in CFT to non-critical systems
- We have looked at QFTs which are 1+1 dimensional and which may be viewed as "massive perturbations" of CFT
- We call these theories integrable models (e.g. sine-Gordon, Lee-Yang theories) and they come with a set of "tools" for computing correlation functions which makes them particularly attractive
- Some of our main results do also hold for generic 1+1 dimensional QFTs [B. Doyon'09]

Short and Long Distance Behaviour

- Recall that [Cardy, OC-A & Doyon'08]:

$$Z_n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn}\frac{Z_n}{Z_1^n}$$

where D_n is a normalisation constant ($D_1 = Z_1$ & $D_1' = 0$), and d_n is the conformal scaling dimension of $\mathcal{T}$ [Knizhnik'87; Calabrese & Cardy'04]:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \sim r^{-2d_n} \ \Rightarrow \ S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n \sim \langle \mathcal{T}\rangle_n^2 = g_n^2 m^{2d_n} \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m\varepsilon) + U$$

with $U = -2g_1'$.

Correlation Functions from Form Factors

- In order to simplify matters let us now think of a QFT with a single particle spectrum. In the n-replica model, there will be n particles that we can label by $j = 1, \ldots, n$
- The two-point function of branch-point twist fields can be decomposed as follows, giving a *large-distance expansion*:

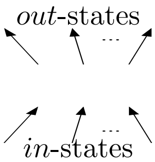
$$\begin{aligned} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\ &= \sum_{\text{state } k} \langle \mathrm{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \end{aligned}$$

where $\sum_k |k\rangle\langle k|$ is a sum over a complete set of states in the Hilbert space of the theory

- The matrix elements $\langle \mathrm{gs}|\mathcal{T}(0)|k\rangle$ are called *form factors*
- For integrable models, an specific program exists (*form factor program*) that allows their exact computation
- However the program needs to be modified to include twist fields correctly

Particles and States in 1+1 dimensional QFT

- In QFT, the Hilbert space is described by particles coming from the far past (*in*-states) or going to the far future (*out*-states). The overlap between *in*- and *out*-states is the *scattering matrix*.



- With particle trajectories chosen to meet all at a point in space-time, the set of all possible configurations of incoming particles (particle types and rapidities) forms a basis for the Hilbert space. Likewise for outgoing particles
- These *in*- or *out*-states are denoted by $|\theta_1, \theta_2, \ldots, \theta_k\rangle^{in,out}_{\mu_1,\mu_2,\ldots,\mu_k}$ with $\theta_1 > \ldots > \theta_k$ for *in*-states and the opposite for *out*-states, where θ_i's are rapidities and μ_i's are particle types

Particles and States in 1+1 dimensional QFT

- Energy and momentum of these states are the sums of those of individual particles: $E = \sum_{i=0}^{k} m_{\mu_i} \cosh\theta_i$ and $P = \sum_{i=0}^{k} m_{\mu_i} \sinh\theta_i$.
- In terms of these states, the generic state $|k\rangle$ in our form factor (FF) expansion is:

$$|k\rangle = |\theta_1, \theta_2, \ldots, \theta_k\rangle^{in}_{\mu_1 \ldots \mu_k}$$

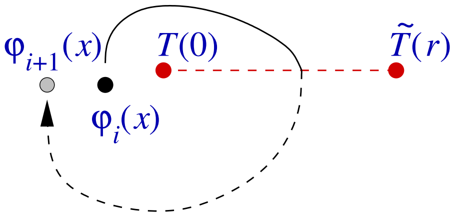
- The quantum numbers $\mu_1, \ldots, \mu_k$ will label the copy number in the replica theory

Form Factor Programme for Twist Fields

- Recall the general commutation relations with the fundamental fields of the replica model:

$$
\begin{aligned}
\varphi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\varphi_{i+1}(y) \qquad x^1 > y^1, \\
\varphi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\varphi_i(y) \qquad x^1 < y^1, \\
\\
\varphi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\varphi_{i-1}(y) \qquad x^1 > y^1, \\
\varphi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\varphi_i(y) \qquad x^1 < y^1.
\end{aligned}
$$

  for $i = 1, \ldots, n$ and $n + i \equiv i$.
- Diagramatically:

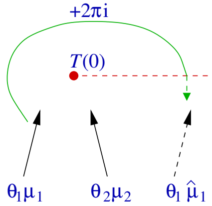


Form Factor Programme for Twist Fields

- Consider an integrable QFT with one particle, no bound state, and S-matrix $S(\theta)$. The Form Factor programme for local fields in integrable QFT was developed long ago [P. Weisz'77; M. Karowski, P. Weisz'78; F.A. Smirnov'92]
- For twist fields [J.L. Cardy, OC-A,B. Doyon'08]
- The S-matrix of the replica theory is $S_{ij}(\theta) = S(\theta)^{\delta_{ij}}$
- The FFs satisfy the monodromy equations:

$$F_k^{\cdots\mu_i\mu_{i+1}\cdots}(\ldots,\theta_i,\theta_{i+1},\ldots) = S_{\mu_i\mu_{i+1}}(\theta_i-\theta_{i+1})F_k^{\cdots\mu_{i+1}\mu_i\cdots}(\ldots,\theta_{i+1},\theta_i,\ldots)$$

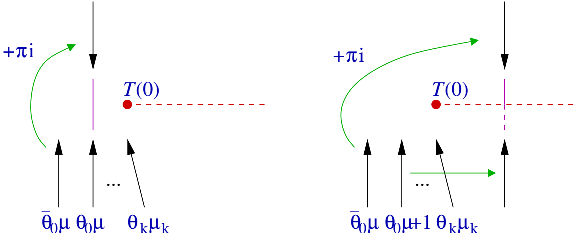
$$F_k^{\mu_1\mu_2\cdots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = F_k^{\mu_2\cdots\mu_k\ \mu_1+1}(\theta_2,\ldots,\theta_k,\theta_1)$$



Form Factor Programme for Twist Fields

- The FFs also satisfy residue equations

$$-i\text{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mu\mu\mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1 \ldots, \theta_k) = F_k^{\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$



$$-i\text{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mu\ \mu+1\ \mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1 \ldots, \theta_k)$$

$$= -\prod_{i=1}^{k} S_{\mu\mu_i}(\theta_{0i}) F_k^{\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$

- These equations can be solved recursively as they relate lower- to higher-particle form factors

Two-Particle Contribution

$$
\begin{aligned}
\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\
&= \sum_{\text{state } k} \langle \mathrm{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\
&= \langle \mathcal{T}\rangle^2 + n\sum_{j=1}^{n}\int d\theta_1 d\theta_2 e^{-mr(\cosh\theta_1+\cosh\theta_2)}|F_2^{1j}(\theta_1-\theta_2)|^2+\ldots \\
&= \langle \mathcal{T}\rangle^2\left(1+\frac{n}{4\pi^2}\int_{-\infty}^{\infty} f(\theta,n)K_0(2mr\cosh(\theta/2)d\theta+\ldots\right)
\end{aligned}
$$

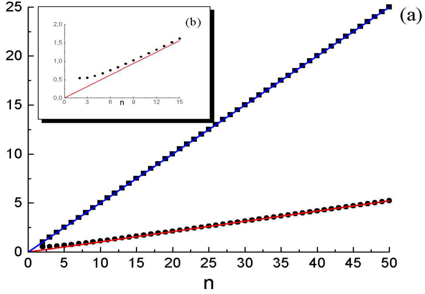
where

$$
f(\theta,n)=\langle \mathcal{T}\rangle^{-2}\sum_{j=0}^{n-1}|F_2^{11}(-\theta+2\pi i j)|^2
$$

- Here we are considering a theory with vanishing one-particle form factor (even if it was non-vanishing it would not change the result for the entropy)
- Main difficulty: analytically continue $f(\theta,n)$ for $n\in\mathbb{R}$, $n\leq 1$, then take the derivative at $n=1$.

Analytic Continuation: Examples

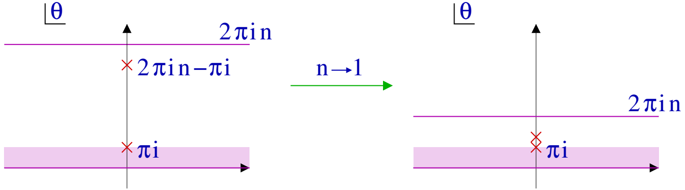
- We need to evaluate $\lim_{n\to 1} \frac{d}{dn}(nf(\theta,n))$
- We need the analytic continuation $\tilde{f}(\theta,n)$ of $f(\theta,n)$ from $n = 1,2,3,\ldots$ to $n \in [1,\infty)$



- Problem: the analytic continuation $\tilde{f}(\theta,n)$ of $f(\theta,n)$ does not converge uniformly as $n \to 1$ on $\theta \in \mathbb{R}$, that is, $\tilde{f}(0,1) \neq f(0,1) = 0$

Analytic Continuation Resolved

- The non-zero value of $\tilde{f}(0,1)$ is due to the collision of poles of $|F_2^{11}(2\pi i j)|^2$ as function of j as $n \to 1$



Analytic Continuation Performed

- Extracting the poles and resuming in j exactly gives:

$$\tilde{f}(\theta,n) \sim_{n\to 1} \tilde{f}(0,1)\left(\frac{i\pi(n-1)}{2(\theta+i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta-i\pi(n-1))}\right)$$

with

$$\tilde{f}(0,1) = \frac{1}{2}$$

- Hence the derivative is supported at $\theta = 0$:

$$\frac{d}{dn}\left(n\tilde{f}(\theta,n)\right)_{n=1} = \pi^2 \tilde{f}(0,1)\delta(\theta)$$

- This gives the universal correction to saturation presented in Benjamin's talk:

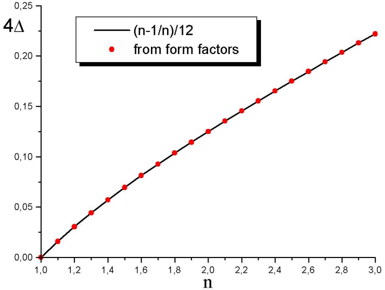
$$-\lim_{n\to 1}\frac{d}{dn}\left(\frac{n}{4\pi^2}\int_{-\infty}^{\infty} f(\theta,n) K_0(2mr\cosh(\theta/2)d\theta + \ldots\right) = -\frac{1}{8}K_0(2mr)$$

Analytic Continuation: General Understanding

- The two-particle example above gives an indication of the sort of difficulties that may be encountered when performing the analytic continuation in n
- This example is nice because it can be resolved in the same way for every 1+1 dimensional QFT (even non integrable [B. Doyon'09])
- There is however no general understanding on how to perform the analytic continuation for higher particle form factors in interacting theories (we completely understood this for the Ising model in [O.C-A & B. Doyon'09])
- The problem is also not fully solved for CFT for more complicated geometries (e.g. several disconnected regions [P. Calabrese, J.L. Cardy & E. Tonni'09])

Consistency Checks

- Given the above it seems difficult to find the right analytic continuation in general
- However, there are some rather strong consistency checks we can perform. A good example can be seen below:



- Here we are examining the short-distance behaviour of the two-point function of twist fields from a FF expansion for the Ising model [O.C-A & B. Doyon'09]

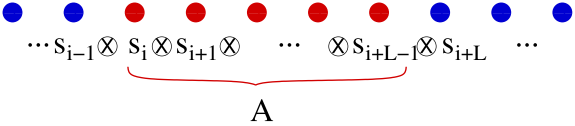
New Predictions for Quantum Field Theory

- We have already seen that a FF computation of the EE allows us to predict *the universal saturation constant as well as exponentially decaying corrections thereof*
- This is a deep result as it changes the usual motivation to study EE. In other words, the EE is not only useful as a means to characterise critical points, but it can also give us *universal information about the near-critical regime described by massive QFT*
- In our most recent work [D. Bianchini, O.C-A & B. Doyon'15] we have also discovered that the EE may allow us to tell unitary from non-unitary critical points apart by examining the leading correction to saturation of the EE For the Lee-Yang theory this is given by

$$S(r) = -\frac{c_{\text{eff}}}{3}\log(m\epsilon) + U - \underbrace{\frac{2}{\pi f(\frac{2\pi i}{3},1)^2}\left(\frac{1}{\sqrt{3}} - \frac{13\pi}{108}\right)}_{0.0769782} K_0(mr) + \cdots$$

Numerical Tests

- Let us go back to our original set-up, that is the bi-partitioned quantum spin chain



- Consider the following two models:

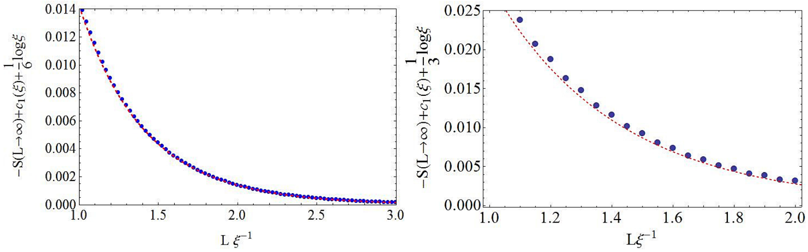
$$H_{\text{Ising}} = -\frac{J}{2}\sum_{i=1}^{N}\left(\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z\right)$$

$$H_{\text{XXZ}} = J\sum_{i=1}^{N}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z\right)$$

- Using either exact diagonalization (for Ising) or DMRG (for XXZ) we may compute the EE for each of these models near (but away from) their critical points [O.C-A, B. Doyon & E. Levi'12]

Bessel Functions and Particle Counting

- The numerical results are shown below:



- In the first figure, numerical results are fitted to the function $1/8K_0(2L/\xi)$ with very good agreement. This is what we would expect for the Ising model
- In the second figure, numerical results are fitted to the function $1/4K_0(2L/\xi)$ with good agreement. This is what we would expect for the sine-Gordon model, which has two fundamental particles of equal mass

Conclusions

- QFT techniques are a powerful tool for predicting the scaling behaviour of the entropy of both critical and non-critical systems
- The EE encapsulates information about the particle spectrum of non-critical theories in 1+1 dimensions
- Many open problems remain in this area which is still dominated by the investigation of critical systems
- A natural next step is to look at other measures of entanglement which are more natural for mixed states (e.g. negativity), consider the EE of excited states and/or disconnected regions